Research note

Recovering a signed Wilson face on four vertices

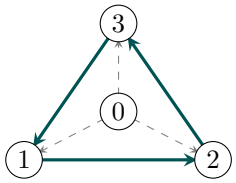
Given six oriented unit-quaternion links on K_4, can a trained classifier recover the sign of the normalized trace around a specified triangle after independent changes of basis at each vertex? We compare eight fitted arms at three seeds, three fixed controls and a continuous stress panel. The experiment separates learning a readout from supplying the correct ordered product as an input.

The target and its symmetry

Let U_{ab} transport a state from vertex a to b. A local frame change sends U_{ab} to $g_b U_{ab} g_a^{-1}$; reverse traversal uses U_{ab}^{-1}. Ordered multiplication left-composes each newly traversed link. The face product based at vertex 1 and its label are

$$H = U_{13}^{-1} U_{23} U_{12}, \qquad h = \tfrac{1}{2} \operatorname{tr} H, \qquad y = \operatorname{sign}(h).$$

The intermediate frames cancel, leaving $H \mapsto g_1 H g_1^{-1}$. This proves invariance of the trace. Reversing the entire walk also preserves the real SU(2) trace. Exchanging two factors at a common basepoint is a different operation and changes the sign on this fixture.



The solid arrows give the target face $1 \to 2 \to 3 \to 1$. The dashed edges form a spanning tree rooted at 0. All six physical links enter the fixture.

Finite training, continuous diagnostic

The finite subgroup $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ supplies six permutations of orthogonal imaginary chord axes. Each permutation has two sign orbits, each with 2,048 distinct gauge presentations. Holding out complete axis permutations assigns 12,288 rows to training, 4,096 to validation and 8,192 to test. The test set therefore has four orbits across two structures. Every individual edge and every pair of edges has matching class-conditional histograms.

The stress set contains 256 pairs of reflected triples of continuous imaginary unit quaternions. Reflection preserves the pairwise dot products and reverses the target sign. Each geometry is presented using two independent local frame draws. These diagnostics have no role in training or checkpoint selection. The fixture contains synthetic mathematical connections; the scores quantify recovery of this constructed observable.

Results

What survives the change of frame?

	Q_8 test		Continuous SU(2) stress		
Method	Accuracy	Worst orbit	Accuracy	Pair rank	Agreement
Raw MLP	100.00	100.00	65.89	73.70	56.64
Edge bag	49.96	20.41	50.78	50.00	61.33
Directed MPNN	49.93	11.25	48.31	49.35	84.51
TokenGT	50.14	1.19	49.54	49.09	97.59
Rooted Gram MLP	50.00	0.00	50.00	50.00	100.00
Oriented MLP	100.00	100.00	97.98	100.00	100.00
Cyclic ququart + affine	100.00	100.00	100.00	100.00	100.00
Dephased + affine	50.00	0.00	50.00	48.83	100.00
Reference traces	50.00	0.00	50.00	50.00	100.00
Ordered Wilson	100.00	100.00	100.00	100.00	100.00
Dephased return	50.00	0.00	49.41	48.83	97.27

Values are percentages. Learned rows average the three registered seeds. Worst orbit is computed within each seed before averaging. Pair rank awards half credit for tied mirror scores. Agreement compares decisions on two frames of the same geometry. A constant classifier can agree perfectly.

The raw MLP and both sign-informed fitted arms attained 100% on the finite test. Under continuous stress, the raw MLP fell to 65.89% with 56.64% frame agreement. The oriented MLP ranked every reflection pair correctly but classified 97.98% of individual presentations at its fixed threshold. The coherent ququart and the classical Wilson product both recovered every sign. The learned dephased head classified 50.00% on both tests. **How to read the comparison.** The orientation arm receives a hand-computed signed volume. The cyclic-ququart arm receives the signed amplitude of a fixed ordered word and learns a scalar calibration. Both have access to an algebraic sufficient statistic. The direct Wilson row evaluates the defining observable. The dephased rows discard off-diagonal coherence after every step. On the finite subgroup this removes the sign of the return amplitude. The generic MPNN and TokenGT receive graph connectivity, edge identity and quaternion coordinates.

Scope. The four held-out finite orbits and the constructed continuous reflection pairs measure a specific kind of transfer. Different presentations of one orbit are dependent. Perfect recovery under an available invariant is compatible with difficulty learning that invariant from coordinates. Favoni and colleagues give a separate lattice architecture for constructing covariant loops with transport and bilinear updates. This experiment tests the prescribed triangle on one fixed graph.

Sources. M. Favoni, A. Ipp, D. I. Müller and D. Schuh, *Lattice gauge equivariant convolutional neural networks*, Phys. Rev. Lett. 128, 032003 (2022). The finite four-vertex witness records the original motivating example and exact checks. The present task concerns a finite graph.