Validated Dirichlet approximation and weighted arithmetic moments

GRH research working note

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Abstract

Adaptive outward dyadic arithmetic and theta-integral quadrature give a requested-error construction for three fixed completed Dirichlet functions. Fresh samples and executed rational readout training certify the supplied uniform 10^{-24} examples on the stated compact; no practical extreme-precision runtime bound is asserted. Separately, a fixed-twist sextic Hecke-family moment improves an original masked remainder estimate, while retaining a positive cutoff loss. We give the precise external analytic inputs and distinguish these results from zero-location and spectral determinant claims. A classical quadratic pivot improves one row sector; joint Gauss phases have an exact masked identity. An ordinary all-character conductor comparison improves the unsigned raw envelope to $H + DH^{1/6} + (DH)^{2/3}$, while an executed finite arithmetic Fourier readout certifies correlation and signed-observable errors without asserting a varying-family signed saving. A moving common-profile construction aligns the actual coefficients on nonprincipal sixth-power rows, and an exact joint example rules out variance-only centering; the fixed-profile signed target remains open. An exact conductor-12 character twist and gcd–Möbius inversion give a separate bound for the signed coprime slice, with the former 7/6 comparison retained as history. A signed common-divisor zeta expansion and a complete-column exact laboratory retain every raw mask; weighted divisor certificates do not close the required asymptotic saving. An exact rough-prime support identity gives an explicit target-budget mask correction; the remaining coherent signed main term is still unestimated. A pinned fixed-order preprint theorem, applied to the physical sixth-power-free radicands, improves the global unsigned critical power to 9/8 and the $A_6 = 1$ signed slice to 13/12. Mean-retaining coherent reductions and conditional scalar-input diagnostics keep the fixed-profile signed target open. No proof of GRH or complete Lean formalization is asserted. Primitive root normalization, all-component finite dual prefixes, actual-cutoff Schur transfer and a full equal-norm localization retain the coherent unequal-norm target as an open problem. An actual Gauss-weighted short-divisor tail admits a physical cubic-matrix estimate. A ramified theta construction now covers the actual fixed finite-character family, and a whole-factor partition bounds the original pure finite dual prefix with normalized power 101/144, retaining every ray, raw zero and finite cutoff. The infinite-frequency complement and full signed assembly remain open; the whole signed and unsigned exponents are unchanged. The paired cutoff analysis gives target-size estimates on stated pure and mixed selections of the signed equality–complement cross, with an exact ordered divisor identity.

1 A finite, validated approximation result

Let $\chi_3 = (0, 1, -1)$, $\chi_4 = (0, 1, 0, -1)$ and $\chi_5 = (0, 1, -1, -1, 1)$, extended periodically, with parities $a = 1, 1, 0$. We approximate the completed functions

$$\Lambda_q(s) = (q/\pi)^{(s+a)/2}\Gamma((s + a)/2)L(s, \chi_q).$$

The primitive theta/Mellin identity is the explicit classical analytic input; no hypothesis about zeros is used. These three examples do not assert a uniform result for all conductors, general Hecke functions or Artin holomorphy.

Theorem 1 (Scope of the supplied finite certificate)**.** *For each $q = 3, 4, 5$, the supplied exact-rational calculation constructs a complex polynomial P_q of degree 28 satisfying*

$$\sup_{|s-1/2|\le 7/10} |P_q(s) - \Lambda_q(s)| < 5 \cdot 10^{-15}.$$

The computation includes enclosures of all 64 true Fourier-circle nodes, completed-function sample error, analytic truncation, Fourier aliasing, the exact closed-form 180-step gradient-descent iterate and coefficient midpoint rounding. The Python implementation and informal analytic proof have independent checks; this is not a kernel-checked Lean theorem.

Proof. Write $F(c+z) = \sum_{n\ge 0} a_n z^n$, with $c = 1/2$, $R = 7/10$, $\rho = 6/5$, $r = 12/5$. The independently recomputed rational theta majorants are strictly below $M_q = 4, 10, 6$, respectively, on $|z| \le r$. Use true roots $\zeta_j = e^{2\pi i j/64}$ and rational labels y_j with proved error $|y_j - F(c + \rho\zeta_j)| \le \eta$. Use the real Euclidean gradient on complex coefficient coordinates for

$$\mathcal{L}(b) = \frac{1}{128} \sum_{j=0}^{63} \left| \sum_{n=0}^{28} b_n \zeta_j^n - y_j \right|^2.$$

Discrete Fourier orthogonality gives the least-squares optimum $b_n^* = 64^{-1} \sum_j y_j \bar{\zeta}_j^n$, $0 \le n \le 28$, and gradient $\nabla\mathcal{L}(b) = b - b^*$. For step $1/4$ and zero initialization, the exact 180-step iterate is $b_n^{180} = (1 - (3/4)^{180}) b_n^*$. The code evaluates this closed form; it does not claim an explicit180-update optimizer run.

Set $u = R/\rho = 7/12$, $v = R/r = 7/24$ and $w = (\rho/r)^{64} = 2^{-64}$. Let $\widehat{b}_n$ be an outward rectangle enclosing b_n^* and ϵ_n the sum of the coordinate halfwidths of the rectangle for b_n^{180}. Choose each actual polynomial coefficient as that rectangle's exact rational midpoint c_n. The normalized basis is explicit:

$$P_q(s) = \sum_{n=0}^{28} c_n \left(\frac{s - 1/2}{6/5} \right)^n.$$

Thus the exported coefficient with index n multiplies the normalized variable, rather than the raw power $(s - 1/2)^n$. Its error on the compact disk is bounded by

$$E_{\text{opt}} = (3/4)^{180} \sum_{n=0}^{28} |\widehat{b}_n|_1 u^n, \qquad E_{\text{arithmetic}} = \sum_{n=0}^{28} \epsilon_n u^n,$$

$$E_{\text{oracle}} = \eta \frac{1 - u^{29}}{1 - u}, \qquad E_{\text{alias}} = M_q \frac{w}{1 - w} \frac{1 - v^{29}}{1 - v},$$

$$E_{\text{Taylor}} = M_q \frac{v^{29}}{1 - v}.$$

Here $|\widehat{b}_n|_1$ is the sum of the maximum absolute real and imaginary coordinates, a safe upper bound for every enclosed complex modulus. Cauchy's estimate $|a_n| \le M_q r^{-n}$ and the Fourier identity $b_n^* = a_n \rho^n + \sum_{\ell \ge 1} a_{n+64\ell} \rho^{n+64\ell}$ for exact labels prove the alias and Taylor bounds. Label perturbations contribute at most η per Fourier coefficient. The remaining two terms follow directly from the exact iterate and coefficient rectangles. Each term is rounded upward to a 180-bit dyadic rational; the reported total is the exact sum of those reported terms. The certified totals below are strictly below the asserted threshold. Detailed sample enclosures are proved in the extended edition and encoded with exact integer endpoints. □

Conductor	Maximum sample radius	Uniform polynomial bound
3	3.026×10^{-17}	1.786×10^{-15}
4	7.206×10^{-17}	4.455×10^{-15}
5	1.404×10^{-16}	2.906×10^{-15}

The displayed values are rounded upward; exact rational records establish the strict inequalities.

2 What this does and does not certify

The final JSON records exact rational centers, interval endpoints, source hashes and all five error terms. Float values are display summaries only. A high-precision comparison is an independent diagnostic, not the reason an interval contains its mathematical target. Arithmetic uses outward integer operations, and invalid float/order inputs are rejected, including through typed caches. Earlier pilots and failed validation probes are retained.

This is a fixed finite tolerance/configuration. Increasing accuracy needs new truncation and precision choices; the fixed $N = 11, T = 128$ interface is not an arbitrary-accuracy oracle. The broader approximation existence theorem is separate. The certificate says nothing about the location of zeros: P_q is not required to be real-rooted or a self-adjoint spectral determinant. The independent arithmetic determinant identity, positivity and original signed short-row/covariance estimates remain open. No GRH proof or complete new Lean4 formalization is claimed.

3 Requested-error readouts from preserved certified labels

The accepted final-v5 sample certificate is preserved unchanged. For each $q = 3, 4, 5$, it supplies $m = 64$ rational complex labels y_j, bounds $|y_j - \Lambda_q(c + \rho\zeta_j)| \le \eta_j$, and outward rectangles enclosing the true roots $\zeta_j = e^{2\pi ij/m}$, where $c = 1/2$, $\rho = 6/5$. The classical primitive theta/Mellin identity and accepted growth bounds $M_q = 4, 10, 6$ on $|s - c| \le r = 12/5$ remain the analytic inputs. This extension accepts an exact rational error request $\varepsilon > 0$ on the fixed compact $|s - c| \le R = 7/10$. It changes the finite readout, while retaining the certified sample errors and the 64-node alias budget. It is not an arbitrary-accuracy theta-sampling interface.

Theorem 2 (Executed rational readout with explicit refusal)**.** *For each supported conductor and rational request, the finite-data selection below either returns a normalized rational polynomial P and an executed, exactly replayable gradient-descent trace with*

$$\sup_{|s-c|\le R} |P(s) - \Lambda_q(s)| \le \varepsilon,$$

or refuses the request with the exact minimum base budget over all 64 *allowed degrees. Refusal concerns attainability by this particular fixed-data majorant; it is not an impossibility theorem for approximation.*

Proof. Let B_n be an outward rectangle enclosing $b_n^* = m^{-1} \sum_j y_j \bar{\zeta}_j^n$, $0 \le n < 64$, computed from the preserved root rectangles. Put $t_n = \mathrm{mid}(B_n)$ and let e_n be the sum of its coordinate halfwidths. These are exact rationals, and $|t_n - b_n^*| \le e_n$. For a selected $d < 64$ execute, using exact rational operations,

$$b_n^0 = 0, \qquad b_n^{k+1} = \tfrac{3}{4} b_n^k + \tfrac{1}{4} t_n, \qquad 0 \le n \le d.$$

This is genuine gradient descent with step $1/4$ for the explicitly specified real Euclidean objective $J(b) = \frac{1}{2}\sum_{n=0}^{d} |b_n - t_n|^2$, whose gradient is $b - t$. All iterates and exact objective values are retained. The output is literally

$$P_k(s) = \sum_{n=0}^{d} b_n^k \left(\frac{s-c}{\rho}\right)^n.$$

Set $u = R/\rho = 7/12$, $v = R/r = 7/24$, $w = (\rho/r)^{64} = 2^{-64}$, $S_d(x) = \sum_{n=0}^{d} x^n$ and $\eta = \max_j \eta_j$. The five budgets are

$$E_{\mathrm{opt}}(k) = \sum_{n=0}^{d} |b_n^k - t_n|_1 u^n, \qquad E_{\mathrm{oracle}} = \eta S_d(u),$$

$$E_{\mathrm{alias}} = M_q \frac{w}{1-w} S_d(v), \qquad E_{\mathrm{Taylor}} = M_q \frac{v^{d+1}}{1-v},$$

$$E_{\mathrm{arithmetic}} = \sum_{n=0}^{d} e_n u^n.$$

Here $|x + iy|_1 = |x| + |y|$. Cauchy's estimate and true-root Fourier orthogonality give the alias and Taylor terms; certified label errors give the oracle term. Triangle inequality applied to $b_n^k - b_n^* = (b_n^k - t_n) + (t_n - b_n^*)$ gives the other two. Thus their exact sum bounds the compact-uniform error.

Define $C_d = E_{\mathrm{oracle}} + E_{\mathrm{alias}} + E_{\mathrm{Taylor}} + E_{\mathrm{arithmetic}}$. Select the smallest $d < 64$ with $C_d < \varepsilon$; if none exists, report $\min_{d<64} C_d$ and refuse. For a selected degree, $E_{\mathrm{opt}}(k) = (3/4)^k \sum_n |t_n|_1 u^n \longrightarrow 0$. Therefore executing updates until $E_{\mathrm{opt}}(k) \le \varepsilon - C_d$ terminates after finitely many steps. The serialized coefficients, budgets, targets and trace use exact rational endpoints; displayed decimal approximations are not proof inputs. □

4 Requested-error theta samples for three fixed conductors

For the primitive characters $\chi_3 = (0, 1, -1)$, $\chi_4 = (0, 1, 0, -1)$, $\chi_5 = (0, 1, -1, -1, 1)$ with parities $a = 1, 1, 0$, retain the classical primitive theta/Mellin identity and root number one already stated above. Put $D = [-1, 2] + i[-4/3, 4/3]$ and $\delta = 3/q$. The following adaptive extension is implemented with independent interval and sample checks; it is an informal analytic theorem, not a Lean result.

Theorem 3 (Finite adaptive sample specification)**.** *For every supported conductor, exact rational $s \in D$, and rational $\eta > 0$, the choices below terminate and give a rational complex midpoint y and exact rational radius $R \le \eta$ with $|y - \Lambda_q(s)| \le R$. The same conclusion holds for a specified true circle node when its root and node enclosures are refined on every precision retry.*

Proof. For $x \ge 0$ define the exact rational bound $E(x) = 2^{-\lfloor x \rfloor}$. Since $e > 2$, $e^{-x} \le E(x)$; this bound is nonincreasing and tends to zero. Let $Q(x) = x^{-1} + 2x^{-2} + 2x^{-3}$ and $C_0 = 1$, $C_1 = (1 + \delta^{-1})/2$. For $N \ge 1$, $n_0 = N + 1$, $J \ge 1$, $T = 2^J$, use

$$B_n = 2C_a Q(\delta n_0^2) \frac{E(\delta n_0^2/2)}{1 - E(\delta n_0)},$$

$$B_t = \sum_{n=1}^{N} 2n^a T^3 E(\delta n^2 T) Q(\delta n^2 T).$$

Here $\delta n_0 \geq 6/5$ ensures a positive denominator; $E(x)$ equals one for $0 \leq x < 1$, so this safeguard is essential. On the four panels of each $[2^j, 2^{j+1}]$, write midpoint m, halfwidth h, $\rho = m/2$, $v = h/\rho \leq 2/9$, and $B(m) = \max\{(m+\rho)^2, (m-\rho)^{-2}, 1\}$. Use the Cauchy remainder

$$B_Q = \sum_{\text{panels}} \sum_{n=1}^{N} \frac{2n^a \, 2h \, 3B(m) E(\delta n^2 m/2) v^{2K+2}}{(2K+3)(1-v^2)}.$$

Choose in turn the first $N \geq 1$, first $J \geq 1$, and first $K \geq 0$ for which their respective budgets are at most $\eta/8$. Gaussian decay, then exponential time decay, then $v \leq 2/9$ prove termination. These are the accepted tail and Cauchy arguments with each true exponential separately replaced by its upper bound E.

There are $P = 4JN$ potential pairs. A nonzero-character pair may be omitted only if its entire two-power contribution has bound

$$b = 4hn^a(m+h)^2 E(\delta n^2 (m-h)) \leq \eta/(8P).$$

Add every omitted bound to B_{discard}, giving $B_{\text{discard}} \leq \eta/8$. Evaluate all other integrated Taylor polynomials by the stated exact coefficient recurrence using outward rectangles. Refine both interval precision and all primitive series remainders until the final rectangle's sum of coordinate halfwidths satisfies $R_{\text{arith}} \leq \eta/4$. For this fixed finite expression all input and primitive intervals shrink; continuity and outward arithmetic imply $R_{\text{arith}} \to 0$. Precision doubling therefore terminates, and the sum of the five raw components is at most $3\eta/4$. Choose an integer output scale B such that $5\,2^{-B} \leq \eta/8$ and round each component upward by $\mathrm{up}_B(x) = 2^{-B}\lceil 2^B x \rceil$. The reported radius is their literal exact sum. With $\mathbf{b} = (R_{\text{arith}}, B_n, B_t, B_Q, B_{\text{discard}})$, it obeys

$$|y - \Lambda_q(s)| \leq \sum_{i=1}^{5} \mathrm{up}_B(b_i) \leq \frac{3\eta}{4} + 5\,2^{-B} \leq \frac{7\eta}{8} < \eta.$$

The output retains exact rational midpoints, rounded components, B, raw interval radius, parameters, input metadata and source hashes. The raw analytic bounds are exactly reconstructible; they need not be serialized as potentially enormous fractions. The output precision varies with the request and introduces no fixed reporting floor. □

This fixed-conductor sample result uses no hypothesis on zeros. It does not remove a downstream fixed-node Fourier alias floor, supply an all-conductor theorem or a positive arithmetic determinant, or prove GRH.

5 An end-to-end adaptive analytic readout

The new driver changes the sample truncations, interval precision, circle size and degree; it does not reuse the frozen labels to assert a smaller error. Its scope is still the three stated primitive quadratic characters and the fixed compact $|s - 1/2| \leq 7/10$.

Theorem 4 (Requested-error construction in the stated finite family)**.** *Given a positive rational ε and $q \in \{3, 4, 5\}$, the specified exact-arithmetic algorithm terminates in the mathematical unbounded-resource model and returns a normalized rational polynomial P with*

$$\sup_{|s-1/2| \leq 7/10} |P(s) - \Lambda_q(s)| \leq \varepsilon.$$

Its sample enclosures, rational optimizer updates and all error budgets are explicit. This is an approximation theorem, with the stated classical theta identity as an input; it supplies no zero-location or positive determinant conclusion and is not a Lean theorem or a practical runtime bound.

Proof. Put $u = 7/12$, $v = 7/24$. Choose $d \geq 0$ so $10v^{d+1}/(1-v) \leq \varepsilon/8$, and choose $m > d$ so

$$10\frac{2^{-m}}{1-2^{-m}}\frac{1}{1-v} \leq \varepsilon/8.$$

The driver rounds m upward to a multiple of eight, which only decreases this bound. Set the requested sample error $\eta = \varepsilon(1-u)/8$. The adaptive sampler gives all m true circle nodes and labels with error at most η. Form their outward Fourier rectangles through degree d, and refine the nodes and finite evaluation arithmetic if their weighted midpoint radii exceed $\varepsilon/8$. This refinement terminates: for these fixed finite d, m, node uncertainty and every finite-expression arithmetic width tend to zero with precision. Merely lifting fixed boxes would not suffice, so the driver regenerates the nodes and samples on this retry.

Choose rational midpoint targets t_n and execute the previously specified quadratic-objective updates until $\sum_{n=0}^{d} |b_n^k - t_n|_1 u^n \leq \varepsilon/4$. Their geometric contraction proves finite termination. The oracle, aliasing, Taylor and Fourier-midpoint terms are each at most $\varepsilon/8$, and the optimization term is at most $\varepsilon/4$. Hence the raw five-term sum is at most $3\varepsilon/4$. Choose a serialization precision B with $5\,2^{-B} \leq \varepsilon/8$, round each term upward, then add the five reported rationals exactly. The printed bound is at most $7\varepsilon/8 \leq \varepsilon$. The polynomial uses $((s-1/2)/(6/5))^n$, and the preceding Fourier/Cauchy proof gives its asserted compact-uniform bound. □

The trace is losslessly encoded by the fixed target vector and a rational scale extracted from each *executed* coefficient state, together with its objective, weighted residual and full-state hash. Every coefficient can be recovered exactly; a separate replay expands the states and checks every update. The producer does not generate its trace from a closed form. The saved objective remains the rational Fourier-midpoint quadratic, with its perturbation from ideal-root least squares explicitly budgeted.

6 An arithmetic-family moment with every original row retained

Let $K = \mathbb{Q}(\sqrt{-3})$, $\mathcal{O} = \mathbb{Z}[\omega]$, and write N for the ideal or element norm. Fix a finite-order Hecke character ν and a finite set S of prime ideals containing those above 6 and those dividing its conductor. For $0 \neq u \in \mathcal{O}$, let κ_u be the character of $K(u^{1/6})/K$ defined using arithmetic Frobenius, and put $\psi_u = \mathrm{prim}(\nu\kappa_u)$. Here prim means the underlying primitive Hecke character: it is formed before introducing any divisor variable or evaluating conductor-dependent zero extensions. The rows below are all actual Eisenstein elements, including units and prime powers supported in S.

The external inputs are abelian Kummer reciprocity and character uniqueness, the primitive Hecke functional equation and its usual vertical growth, Stirling's formula, scalar Dedekind-zeta convexity, and Blomer–Goldmakher–Louvel's large sieve for their specified sixth-order family. We use the pinned arXiv:1112.1650v1, Theorem 1.3, Lemma 2.1 and Appendix II; its Section 5 supplies the all-ideal decomposition whose fixed-twist proof is redone here. No statement about all primitive characters of comparable conductor, Artin holomorphy, Lindelöf bounds or zero locations is assumed.

Theorem 5 (The original-family numerator moment)**.** *For every $\varepsilon > 0$, uniformly for $H \geq 1$ and $t \in \mathbb{R}$,*

$$\sum_{0<Nu\leq H} |L(1/2+it, \psi_u)|^2 \ll_{\nu,S,\varepsilon} H^{1+\varepsilon}(1+|t|)^{1+\varepsilon}. \tag{1}$$

For every fixed $0 < \delta < 1/2$, the same estimate holds with $1/2 + it$ replaced by $1/2 + \delta + it$, with the constant also depending on δ. If the central bound has constant C_0 at an exponent $0 < \varepsilon_0 \le 1/4$, an admissible off-central constant is

$$200C_0 \left(\frac{3/2 + \delta}{1/2 - \delta}\right)^2. \tag{2}$$

Proof. For the canonical BGL character $\vartheta_{\mathfrak{a}}$, Appendix II gives a radicand $\alpha_{\mathfrak{a}}$ with $(\alpha_{\mathfrak{a}})_S = \mathfrak{a}\mathfrak{g}^{-6}$. Since $\mathcal{O}_S$ is principal, comparison with any literal u whose outside-S ideal is $\mathfrak{a}$ gives $\kappa_u = \eta\vartheta_{\mathfrak{a}}$ for one of finitely many fixed S-unit Kummer twists η. This is an equality of primitive characters, established outside a finite set and extended by uniqueness. For each outside-S ideal $\mathfrak{a}$ and S-supported integral ideal $\mathfrak{b}$, at most six elements generate $\mathfrak{a}\mathfrak{b}$. The resulting S-supported multiplicity sum converges at exponent $1 + \varepsilon$.

For each fixed $\xi = \nu\eta$, split the canonical parameter as $\mathfrak{a} = \mathfrak{a}_0\mathfrak{a}_1\mathfrak{a}_2^6$, where $\mathfrak{a}_0$ is squarefree, $\mathfrak{a}_1$ is squarefull and sixth-power-free, and $(\mathfrak{a}_0, \mathfrak{a}_1) = 1$. The primitive conductor is at most a fixed multiple of $N(\mathfrak{a}_0\mathfrak{a}_1)$. The common-gamma approximate functional equation, with $X = H/(N\mathfrak{a}_2)^6$ and $T = 1 + |t|$, has common maximal length $B = X^{1/2}T(HT)^{\eta}$ for any sufficiently small fixed $\eta > 0$. Mellin inversion removes its row-dependent conductor weight. Split a column as $\mathfrak{b}_S\mathfrak{b}_1\mathfrak{b}_2$, with $\mathfrak{b}_S$ supported in S, $\mathfrak{b}_1$ squarefree and $\mathfrak{b}_2$ squarefull, the last two coprime and prime to S. The original primitive values at $\mathfrak{b}_S$ remain bounded row multipliers. The S-supported and squarefull column weights are summable at $1/2 + \eta$. Multiplying squarefree column coefficients by the fixed ξ preserves their squared norm, whatever its finite order. BGL's large sieve therefore bounds the energy for fixed $\mathfrak{a}_1, \mathfrak{a}_2$ by

$$(HT)^{O(\eta)}\{M + B + (MB)^{2/3}\}, \qquad M = X/N\mathfrak{a}_1.$$

Squarefull counting and summation give $X + B\sqrt{X} + (XB)^{2/3} \ll (HT)^{O(\eta)}XT$. The $\mathfrak{a}_2$ sum converges at exponent 6. Principal rows are separated before the pole-free functional equation and bounded by scalar zeta convexity. This proves (1).

For the off-central statement divide each $L(s, \psi)$ by $(s + 1)^{5/4}$ and, on principal rows, also multiply by $(s - 1)/(s + 1)$. These are bounded analytic functions on $\operatorname{Re} s > 1/2$, continuous on the boundary. Poisson representation and Jensen's inequality transfer the central squared moment. The convolution of $P_\delta(y) = \delta/[\pi(\delta^2 + y^2)]$ with $(1 + |t|)^{-3/2+\varepsilon_0}$ is at most eleven times that weight when $0 < \varepsilon_0 \le 1/4$. Restoring the denominator and the principal pole-removal factor gives (2). The extended edition gives the detailed derivation with all primitive local values. □

Separate fixed-unitary extension. The moment theorem also holds for any one fixed unitary Hecke character ν. For precision choose $\nu(\mathfrak{a}) = \nu_0(\mathfrak{a})(N\mathfrak{a})^{-i\tau}$, with angular parameter $m \in \mathbb{Z}$ and completed gamma factor

$$\gamma_\nu(s) = 2(2\pi)^{-(s+|m|/2+i\tau)}\Gamma(s + |m|/2 + i\tau).$$

All finite Kummer twists have this same infinity type. Use $T_\nu = 1 + |t + \tau| + |m|$, comparable to $1 + |t|$ with ν-dependent constants. Pure norm rows have $L(s, \psi) = \zeta_K(s + i\tau)$ and are separated before the pole-free functional equation; their pole at $1 - i\tau$ is removed by $(s - 1 + i\tau)/(s + 1 + i\tau)$. No uniformity in m, τ or the conductor of ν is asserted. This extension of the moment is separate from the following finite-order type-I residue formula.

6.1 The exact masked remainder and the remaining loss

Retain the original zero-extended symbol $\chi_d(u) = (u/d)_6$ on good squarefree ideals. Let μ be the ideal Möbius function and set

$$\beta_U(d) = \sum_{\substack{ab=d \\ Na, Nb \le U}} \mu(a)\mu(b), \quad T(u,d) = S \cup \operatorname{supp}(u) \cup \operatorname{supp}(d).$$

Let $\psi_{u,2} = \operatorname{prim}(\psi_u^2)$, formed independently of squaring any zero sentinel. With $\mathfrak{f}_u$ and $\mathfrak{f}_{u,2}$ the primitive conductors, the full-mask series is

$$F_{u,d}(s) = \frac{L(s,\psi_u)}{L(2s,\psi_{u,2})} C_{T(u,d)}(s), \tag{3}$$
$$C_T(s) = \frac{\prod_{p \in T,\ p \nmid \mathfrak{f}_u} (1 - \psi_u(p)(Np)^{-s})}{\prod_{p \in T,\ p \nmid \mathfrak{f}_{u,2}} (1 - \psi_{u,2}(p)(Np)^{-2s})}.$$

Initially it equals $\sum_{c \text{ squarefree},\ (c,T)=1} \psi_u(c)(Nc)^{-s}$ for $\operatorname{Re} s > 1$. Every prime of T remains deleted, including those forgotten by either primitive conductor.

For finite-order ν, $D \ge 1$, $U \ge 1$, $U^2 \le D$, and $W \in C_c^\infty((0,\infty))$ supported in $[1,2]$, put

$$\widehat{W}(s) = \int_0^\infty W(x) x^{s-1}\, dx, \qquad \mathcal{M}_4(W) = \sum_{j=0}^{4} \sup_x |(x\partial_x)^j W(x)|.$$

On $\sigma = 1/2 + \delta$, the exact residue-subtracted type-I remainder is

$$E_u = \frac{D^\delta}{2\pi} \int_{\mathbb{R}} \widehat{W}(\sigma + it) D^{it} \sum_{\substack{d \text{ squarefree} \\ (d,S)=1}} \frac{\beta_U(d)\nu(d)\chi_d(u)}{(Nd)^{\sigma+it}} F_{u,d}(\sigma + it)\, dt. \tag{4}$$

This definition retains the actual β_U, all phases, d-masks, conductors and the independently primitive square.

Theorem 6 (An improved masked residual estimate)**.** *Under the preceding finite-order hypotheses, for every $\varepsilon > 0$,*

$$\sum_{0 < Nu \le H} |E_u|^2 \ll_{\nu,S,\delta,\varepsilon} \mathcal{M}_4(W)^2 D^{2\delta} U^{2-4\delta} H (HD)^\varepsilon. \tag{5}$$

Proof. The same deleted set in both factors of (3) gives $|F_{u,d}(\sigma + it)| \ll (Nu\, Nd)^\eta |L(\sigma + it, \psi_u)|$: the reciprocal incomplete Euler product at $2\sigma > 1$ is bounded by $\zeta_K(2\sigma)$, and its remaining finite numerator is bounded by a small power of its radical. Apply Minkowski to (4) in the exact row space and use Theorem 5. The actual coefficient sum is at most

$$\sum_d \frac{|\beta_U(d)|}{(Nd)^{\sigma-\eta}} \le \left(\sum_{Na \le U} (Na)^{-\sigma+\eta} \right)^2 \ll U^{1-2\delta+2\eta}.$$

Four Mellin integrations by parts bound the remaining height integral by a constant times $\mathcal{M}_4(W)$. Square the result and absorb the small losses using $U^2 \le D$. □

At $H = D^{3/4}$ and $U = D^a$ with fixed $0 < a \le 1/2$, the upper bound's excess over H is $2\delta + (2 - 4\delta)a$, up to small losses; as $\delta \downarrow 0$ it still leaves $2a$. This is a limitation of the upper bound, not an energy lower bound. The extracted principal residue, the original short term at smaller

scales and the signed type-II minus type-I covariance are not controlled by (5). The next arithmetic task is to estimate the complete d-weighted moment before its absolute coefficient sum, or establish sufficient signed covariance. No positive arithmetic determinant, zero-location theorem or GRH proof follows. The analytic argument uses the external classical inputs above; finite diagnostics and selected existing Lean local identities do not formalize the Hecke functions, approximate functional equation, Poisson transfer or these moment bounds.

7 Recombining the actual weighted cutoff

Keep the finite-order Eisenstein family and original masks of Section 6. For $U \geq 1$ and a squarefree ideal m, put

$$A_U(m) = \sum_{d|m} \beta_U(d) = \sum_{\substack{ab|m \\ (a,b)=1,\ Na,Nb\leq U}} \mu(a)\mu(b).$$

Every summation variable here is an integral ideal. The last formula uses squarefreeness of m; an unconstrained square of a Möbius polynomial would also include forbidden overlapping supports.

Theorem 7 (Exact recombination and an absolute-half-plane rate)**.** *Write $T_0 = S \cup \mathrm{supp}(u)$ and $J_{u,U}(s) = \sum_{d \text{ squarefree},\ (d,S)=1} \beta_U(d)\nu(d)\chi_d(u)(Nd)^{-s}F_{u,d}(s)$. For $\mathrm{Re}\, s > 1$,*

$$J_{u,U}(s) = \sum_{m \text{ squarefree},\ (m,T_0)=1} A_U(m)\psi_u(m)(Nm)^{-s},$$

$$|A_U(m)| \leq 3^{\omega(m)}, \qquad A_U(m) = \mu(m) \quad (Nm \leq U).$$

For every $\kappa > 0$ and $0 < \epsilon < \kappa$, uniformly for $\mathrm{Re}\, s \geq 1 + \kappa$, all original rows and all $U \geq 1$,

$$\left| J_{u,U}(s) - \frac{1}{L^{T_0}(s,\psi_u)} \right| \leq U^{-\kappa+\epsilon}\{\zeta_K(1+\epsilon)^3 + \zeta_K(1+\epsilon)\}.$$

Here L^{T_0} removes every prime of T_0, including conductor-drop primes; the bound is in the absolute-convergence half-plane only.

Proof. On a surviving term $m = dc$, the two squarefree ideals c, d are coprime, and both avoid T_0. The original product $\nu(d)\chi_d(u)\psi_u(c)$ is then $\psi_u(m)$. Grouping by m gives A_U. Each prime of m has three possible positions in $(a, b, m/(ab))$, so the absolute sum is at most $3^{\omega(m)}$. When $Nm \leq U$ all divisors fit both cutoffs; the primewise sum is $1 - 1 - 1 = -1$, giving $\mu(m)$.

The same bound justifies absolute convergence and regrouping for $\mathrm{Re}\, s > 1$, because on squarefree ideals $3^{\omega(m)} = \tau_3(m)$ and $\sum_m \tau_3(m)(Nm)^{-v} = \zeta_K(v)^3$ for $v > 1$, where this latter sum runs over all nonzero integral ideals. Subtract the absolutely convergent Möbius series for $1/L^{T_0}$. Only $Nm > U$ remain. Rankin's inequality with exponent $\eta = \kappa - \epsilon$ bounds their absolute sum by

$$U^{-\eta} \sum_m \{\tau_3(m) + 1\}(Nm)^{-1-\epsilon},$$

which proves the stated constant independently of deleted primes, row multiplicities or character phases. □

For W supported in $[1, 2]$, the same exact regrouping gives the original normalized type-I polynomial

$$I_u = D^{-1/2} \sum_{m \text{ squarefree},\ (m,S)=1} A_U(m)\nu(m)\chi_m(u)W(Nm/D).$$

This formula retains all original zero sentinels. It permits a direct family large sieve before any absolute d sum; its quantitative moment is developed separately below. The displayed absolute-half-plane rate alone is not that critical-strip moment.

8 A cutoff-uniform envelope for the original masked remainder

Fix $K = \mathbb{Q}(\omega)$, $\mathcal{O} = \mathbb{Z}[\omega]$, a finite-order Hecke character ν, and a finite set S containing the primes above 6 and the conductor of ν. All unspecified ideal sums below are over nonzero integral ideals. Let $\chi_m(u) = (u/m)_6$ be the original symbol, zero when $(m,u) > 1$, and put

$$\beta_U(d) = \sum_{\substack{ab=d \\ Na, Nb \le U}} \mu(a)\mu(b), \qquad A_U(m) = \sum_{d \mid m} \beta_U(d).$$

The columns d, c, m are squarefree and prime to S where indicated. Retain the primitive characters $\psi_u = \mathrm{prim}(\nu\kappa_u)$, $\psi_{u,2} = \mathrm{prim}(\psi_u^2)$, and the full deleted set $T(u,d) = S \cup \mathrm{supp}(u) \cup \mathrm{supp}(d)$. For $H, D, U \ge 1$, $U^2 \le D$, and $W \in C_c^\infty((0,\infty))$ supported in $[1,2]$, the original normalized type-I piece is

$$I_u = D^{-1/2} \sum_{\substack{m \text{ squarefree} \\ (m,S)=1}} A_U(m)\nu(m)\chi_m(u) W(Nm/D). \tag{6}$$

Its exact principal residue is

$$P_u = \mathbf{1}_{\psi_u=1} \sqrt{D}\, \widehat{W}(1) \sum_{\substack{d \text{ squarefree} \\ (d,S)=1}} \frac{\beta_U(d)\nu(d)\chi_d(u)}{Nd} \rho_{T(u,d)},$$

$$\rho_T = \frac{\mathrm{res}_{s=1}\, \zeta_K(s)}{\zeta_K(2)} \prod_{p \in T} (1 + (Np)^{-1})^{-1}. \tag{7}$$

Define $E_u = I_u - P_u$; this is the same residue-subtracted remainder as the fully masked Mellin contour, with no cutoff limit or profile supremum.

Theorem 8 (Uniformity in the actual truncated cutoff)**.** *For every $\varepsilon > 0$,*

$$\sum_{0 < Nu \le H} |E_u|^2 \ll_{\nu,S,\varepsilon} \|W\|_\infty^2 (HD)^\varepsilon \{H + DH^{1/3} + (HD)^{2/3}\}. \tag{8}$$

The sum is over every actual nonzero Eisenstein element, including units, S-supported powers and sixth-power rows. The constant is independent of U under $U^2 \le D$. This is an application of the classical sextic large sieve with exact original coefficients, not a zero-location theorem.

Proof. On an absolutely convergent line, expand the restored squarefree series $F_{u,d}$. Its condition $(c, T(u,d)) = 1$ includes $(c,d) = 1$, and the original symbols multiply. Grouping $m = cd$ gives (6) by Mellin inversion. Each prime of m may belong to the first truncated Möbius divisor, to the second, or to neither. Consequently $|A_U(m)| \le 3^{\omega(m)} \ll_\eta (Nm)^\eta$, uniformly in U. The length-D coefficient sequence in (6) therefore has squared norm $\ll_\eta \|W\|_\infty^2 D^\eta$.

The all-ideal row norm in Section 3 of Blomer–Goldmakher–Louvel, arXiv:1112.1650v1 gives

$$B_2^1(X, D) \ll_\eta (XD)^\eta \{X + DX^{1/3} + (XD)^{2/3}\}.$$

For completeness, split $a = a_1 a_2^2 \cdots a_5^5 a_6^6$ dyadically; $a_1, \dots, a_5$ are squarefree and pairwise coprime. If their dyadic sizes are A_j, put $R = \prod_{j=1}^6 A_j$ and $M = \max(A_1, A_2)$. Pivot on the first or second variable, using the squarefree powered sieve proved simultaneously in that paper's Section 3.2. The other variables are bounded column multipliers. The block energy is at most

$$(XD)^\eta \{R + DR/M + D^{2/3}R/M^{1/3}\} \sum_m |\lambda_m|^2.$$

The inequalities $R \le X$, $R^3 \le XM^3$ and $R^3 \le X^2M$ follow from $\prod_j A_j^j \le X$ and the pivot choice. Summing the logarithmically many blocks proves the displayed bound.

Finite S-unit and ray-class partitions convert the original literal symbol into the canonical column value $\vartheta_m(a)$ with fixed modulus-one coefficient multipliers. On noncoprime pairs both values are zero: squarefree m has outside-S conductor exactly m. This orientation preserves sixth-power zeros. At most six elements generate each outside-S ideal times an S-supported ideal; the S-supported sums converge at powers 1, 1/3 and 2/3. Thus (8) holds first with I_u in place of E_u.

A principal ψ_u requires every outside-S valuation of u to be divisible by 6. There are $O_{\nu,S}(H^{1/6})$ such literal rows. Equation (7), $|\rho_T| \ll_K 1$, and the actual coefficient sum

$$\sum_d \frac{|\beta_U(d)|}{Nd} \le \left(\sum_{Na\le U} \frac{1}{Na}\right)^2 \ll \log^2(2U)$$

give $\sum |P_u|^2 \ll DH^{1/6}\|W\|_\infty^2 \log^4(2U)$. This is absorbed by the right side of (8), using $U^2 \le D$ and a smaller internal exponent. Finally $|E_u|^2 \le 2|I_u|^2 + 2|P_u|^2$. The extended proof specifies the canonical-family transfer and the powered-sieve input precisely. □

Combining this with the previously proved fixed-twist AFM transfer yields

$$\sum_{0<Nu\le H} |E_u|^2 \ll_{\nu,S,\delta,\varepsilon} \mathcal{M}_4(W)^2 (HD)^\varepsilon \min\{D^{2\delta}U^{2-4\delta}H,\ H + DH^{1/3} + (HD)^{2/3}\}. \tag{9}$$

At $H = D^{3/4}$ and $U = D^a$, $0 \le a \le 1/2$, the excess over H is at most $\min\{2\delta + (2-4\delta)a, 1/2\}$, up to arbitrarily small losses. The new envelope improves the old one for $a > 1/4$ as $\delta \downarrow 0$, but still leaves an exponent loss of 1/2. It does not settle the original signed short-row bound, type-II covariance, positive arithmetic determinant comparison or GRH. The principal-residue bound used here is sufficient for this envelope, not the stronger target needed to reassemble the full signed argument.

9 High cutoffs, actual bilinear factors and a signed covariance test

Retain the original Eisenstein family, fixed finite-order ν, fixed S, common $W \in C_c^\infty((0,\infty))$ supported in $[1,2]$, and every raw shared-prime zero. Put

$$C_U(m) = \sum_{\substack{abc=m \\ Na>U,\, Nb>U}} \mu(a)\mu(b), \qquad r_U(m) = \#\{(a,b) : ab = m,\ Na, Nb > U\}.$$

All factors are ideals, squarefree and pairwise coprime; on the active annulus $A_U = C_U - \mu$. A coefficient identity is different from a row-energy bound.

Proposition 9 (The high-cutoff window)**.** *If $D > 1$ and $2D/7 \le U^2 \le D$, then for every surviving squarefree column $D \le Nm \le 2D$,*

$$C_U(m) = \mu(m) r_U(m), \qquad A_U(m) = \mu(m)(r_U(m) - 1), \qquad |A_U(m)| \ge 1.$$

For each fixed nonzero W, the normalized squared coefficient norm satisfies

$$D^{-1} \sum_{\substack{m \text{ sf} \\ (m,S)=1}} |A_U(m) W(Nm/D)|^2 \gg_{W,S} 1 \tag{10}$$

for large D, uniformly in this window. This is not a lower bound for row energy.

Proof. The least norm of a nonunit ideal outside S is at least 7. A long-long term with $c \neq 1$ would have $Nm > 7U^2 \geq 2D$. Thus $ab = m$, including both endpoints. The fixed-point-free swap $(a, b) \mapsto (b, a)$ makes r_U even, so A_U is an odd nonzero integer. The last claim follows by counting squarefree ideals on a fixed interval where $|W|$ is bounded below; the constant and error are proved in the extended proof. □

The exact remaining polynomial is genuinely bilinear, with $(a, b) = 1$, $U < Na, Nb < 2D/U$, $D \leq Na\,Nb \leq 2D$, and $x_{ab}(u) = x_a(u)x_b(u)$, including zeros. Both factors see the same row $u = \varepsilon\alpha\beta^2$. Cauchy followed by the accepted powered BGL bound gives only

$$Q_{B_X}(C_U, C_U) \ll_{\nu,S,\epsilon} (DX)^{\epsilon}\|W\|_{\infty}^2\{\sqrt{D}\,X^2 + DX + D^{5/6}X^{5/3}\}. \tag{11}$$

At $X \asymp D^{1/4}$ the exponents are $1, 5/4, 5/4$; the earlier regrouped bound has $1/2, 5/4, 13/12$ and is stronger. No BT_σ saving is obtained. Multiplying two marginal operator norms does not supply a coupled fourth moment.

A finite test in the original family. There is no automatic nonnegative signed covariance, even when $C_U(m)\mu(m) \geq 0$. Take $\nu = 1$, S the primes above $2, 3$, $D = 400$, $U = 17$, and all 72 literal rows of B_7. Choose the common nonnegative smooth profile constructed in the extended proof, with values 10 at 439/400, 1 at 475/400, and zero at every other integer norm divided by 400. There are exactly four active squarefree ideal columns, of norms $439, 439, 475, 475$. In that order,

$$A = (1, 1, 1, 1), \quad C = (0, 0, 2, 2), \quad \mu = (-1, -1, 1, 1), \qquad \Re G_{B_7} = \begin{pmatrix} 72 & 0 & 0 & 18 \\ 0 & 72 & 18 & 0 \\ 0 & 18 & 72 & 0 \\ 18 & 0 & 0 & 72 \end{pmatrix}.$$

Exact raw-symbol enumeration, independently reproducible from the ideal and residue-field definitions, gives

$$Q(C, \mu) = -\frac{27}{25}, \qquad Q(C, C) = \frac{36}{25}, \quad Q(\mu, \mu) = \frac{864}{25}, \quad Q(A, A) = \frac{954}{25}. \tag{12}$$

The real Gram matrix is positive definite, with eigenvalues $54, 90$; its positivity does not force a cross covariance to be positive. This fixed finite example refutes that shortcut, not the asymptotic BT_σ target or the original signed mean-square estimate.

The two-cutoff contour remains coupled. For $T_0 = S \cup \mathrm{supp}(u)$, $\psi = \psi_u$ and independently primitive $\psi_2 = \mathrm{prim}(\psi^2)$, put $q_p = \psi(p)(Np)^{-s}$, $\alpha_p = (Np)^{-z}$, $\beta_p = (Np)^{-w}$ outside T_0. The actual three-placement Euler factor is $1 + q_p - q_p\alpha_p - q_p\beta_p$. Its exact factorization is

$$\mathcal{F}(s, z, w) = \frac{L^{T_0}(s, \psi)}{L^{T_0}(s + z, \psi)L^{T_0}(s + w, \psi)} \prod_{p \notin T_0} \left(1 - \frac{q_p^2(1 - \alpha_p)(1 - \beta_p)}{(1 - q_p\alpha_p)(1 - q_p\beta_p)}\right). \tag{13}$$

Exact double Perron inversion is valid on positive auxiliary lines $c, d > 1 - \Re s$ for $1/2 < \Re s < 1$, retaining the original principal pole exclusion. These safe lines reproduce the previous $U^{2-4\delta}$ energy loss at $\Re s = 1/2 + \delta$, with the same common-profile derivative and fixed-margin dependencies. The correction product alone does not justify moving those lines through reciprocal-L poles. In fact $\mathcal{F}(s, z, 0) = 1/L^{T_0}(s + z, \psi)$ as a meromorphic identity. No zero-free region, reciprocal moment, analytic saving, new Lean formalization or GRH proof is supplied by this construction.

10 A quantitative covariance budget without a new exponent

Retain the original Eisenstein sextic symbols, fixed finite-order ν, fixed S, one possibly complex common $W \in C_c^\infty$ supported in $[1,2]$, all squarefree columns prime to S, and every raw shared-prime zero. In the accepted window $2D/7 \le U^2 \le D$, put $r_m = r_U(m)$, $\lambda_m = \mu(m)\nu(m)W(Nm/D)$ and $v_m = \overline{\lambda_m}$. Thus $C_U = \mu r$ and $A_U = \mu(r-1)$ on the active annulus. For a finite literal row block B, let

$$G_{mn} = \sum_{u \in B} x_m(u)\overline{x_n(u)}, \qquad R = \operatorname{diag}(r_m), \qquad M_j = \sum_m r_m^j |W(Nm/D)|^2 \quad (0 \le j \le 2).$$

The primitive $\psi_u, \psi_{u,2}$ and full deleted sets retain their previous meanings; no conductor-drop value replaces a raw zero.

Proposition 10 (An actual-vector lower budget)**.** *If $0 \le G \le \Lambda_B I$, then*

$$\Re Q_B(C_U, \mu) \ge -\frac{\Lambda_B}{2D}\{\sqrt{M_0 M_2} - M_1\}. \tag{14}$$

In this high window, uniformly in U,

$$\sum_{D \le Nm \le 2D} r_m \ll_K D, \qquad \sum_{D \le Nm \le 2D} r_m^2 \ll_K D\log(2D). \tag{15}$$

Consequently

$$\Re Q_B(C_U, \mu) \ge -C_K \|W\|_\infty^2 \Lambda_B \sqrt{\log(2D)}.$$

The vector inequality is sharp among positive contractions with the specified v, Rv; no such sharpness is asserted for the arithmetic Gram. The moment bound counts actual balanced ideal factorizations, rather than assuming independence of the two row factors.

On B_X, inserting the previously accepted powered-BGL norm gives only

$$\Re Q_{B_X}(C_U, \mu) \ge -C_{\nu,S,\epsilon}\|W\|_\infty^2 (DX)^\epsilon \{X^2 + DX + D^{2/3}X^{5/3}\}.$$

At $X \asymp D^{1/4}$ the exponents remain $1/2, 5/4, 13/12$. At $H = D^{3/4}$ this budget still has the old $D^{1/2}$ excess over H; it supplies no BT_σ saving. A negative-part budget is weaker than $2\Re Q(C_U, \mu) \ge Q(C_U, C_U) + Q(\mu, \mu) - R_\sigma$.

For $S_R = \sqrt{R}$, the exact identity

$$\tfrac{1}{2}(RG + GR) = S_R G S_R + \tfrac{1}{2}[S_R, [S_R, G]]$$

localizes the commutator to unequal factor-count levels, with entry $(\sqrt{r_m} - \sqrt{r_n})^2 G_{mn}$. If B retains all six literal units, the actual unit sum further gives $G_{mn} = 0$ unless $Nm \equiv Nn \pmod{36}$. These are exact deletions, not a power saving. Even zero different-level pairings leave the nonnegative within-level energies $\sum_k (k-1)^2 Q_B(\mu \mathbf{1}_{r=k}, \mu \mathbf{1}_{r=k})$. The unequal-product coupled fourth moment and its mixed Möbius term remain unestimated. No new analytic exponent, energy lower bound, Lean formalization or GRH conclusion follows.

11 The fourth-moment conductor and its full raw zero set

Retain the original Eisenstein sextic symbols $x_m(u) = (u/m)_6$, fixed finite-order ν, fixed S, and one possibly complex common smooth W supported in $[1,2]$. Put $\lambda_m = \mu(m)\nu(m)W(Nm/D)$ and $r_m = r_U(m)$. Columns remain squarefree and prime to S. In $2D/7 \le U^2 \le D$, the actual polynomial is

$$\mathcal{C}_U(u) = \sum_{\substack{(a,b)=1,\ Na,Nb>U \\ D\le Na\,Nb\le 2D}} \mu(a)\mu(b)\nu(ab)W(Nab/D)x_a(u)x_b(u).$$

Its squared row energy has four factor indices a, b, c, d, with $(a, b) = (c, d) = 1$. These internal masks are essential. At each prime,

$$e_p = \mathbf{1}_{p|a} + \mathbf{1}_{p|b} - \mathbf{1}_{p|c} - \mathbf{1}_{p|d} \quad \text{belongs to } \{0, \pm 1\}.$$

Thus, for the two squarefree products $m = ab, n = cd$, the primitive oscillatory conductor outside fixed S is $q = mn/(m, n)^2$. The full raw zero-preserving period is $L = mn/(m, n)$:

$$x_m(u)\overline{x_n(u)} = \mathbf{1}_{(u,(m,n))=1}\theta_{m/(m,n),\,n/(m,n)}(u).$$

No common-prime zero is restored by taking the primitive character.

A generic fourth moment $|\sum_m \lambda_m x_m(u)|^4$ has four individually squarefree columns $m_1, \dots, m_4$ with no such internal coprimality. Its signed local exponent $e_p = \sum_{i=1}^{2} \mathbf{1}_{p|m_i} - \sum_{i=3}^{4} \mathbf{1}_{p|m_i}$ can also be ± 2, giving a nontrivial cubic component. For $L = \mathrm{rad}(m_1m_2m_3m_4)$, the exact row-power interface is

$$Q_j = \prod_{\substack{p|L \\ je_p\not\equiv 0 \ (6)}} p, \qquad Z_j = L/Q_j, \qquad \prod_{i=1}^{2} x_{m_i}(v^j)\overline{\prod_{i=3}^{4} x_{m_i}(v^j)} = \theta_{j,e}(v)\mathbf{1}_{(v,Z_j)=1}.$$

Here $\theta_{j,e}$ is primitive on Q_j, with its own zero extension. At $j = 3$ an exponent ± 2 becomes primitive-principal but its raw zero remains. For the original two-squarefree-product expansion, $Q_j = q$ for all $1 \le j \le 5$, including the quadratic $j = 3$ direction. For the accepted row blocks $Nr_j \asymp A_j$, put $R_0 = \prod_j A_j$, $P = \max(A_1, \dots, A_5)$ and retain $\prod_j A_j^j \le H$. The full A_3 saturation face and all row-factor masks remain.

The small-conductor pair count recovers an already controlled high-gcd sector; it is not a new analytic advance. On the remaining original sector $Nq > P^2$, favorable smooth completion costs $\min(A_j, \sqrt{Nq}) = A_j$ for $1 \le j \le 5$. The literal sharp cutoff is not replaced by this smooth trial. The signed large-conductor fourth-moment and mixed Möbius sums remain open; no improved full exponent, BT_σ, new Lean theorem, zero-location result or GRH conclusion follows.

12 A quadratic pivot on the cube factor

Retain the original zero-extended sextic symbols $\chi_m(u) = (u/m)_6$ over $K = \mathbb{Q}(\omega)$, the fixed S containing all primes above 6 and comparison moduli, and the squarefree good columns $D \le Nm \le 2D$. Write an outside-S row ideal uniquely as $r_1r_2^2r_3^3r_4^4r_5^5r_6^6$: only $r_1, \dots, r_5$ are pairwise-coprime squarefree ideals; r_6 is arbitrary and may overlap them. Use primary generators. Fix one literal unit and one S-supported row factor, retaining their original values. Let $A_j \le Nr_j < 2A_j$, $A_j \ge 1$, $R_0 = \prod_j A_j$ and $\prod_j A_j^j \le H$; the actual capacity restriction on rows is retained. Coefficients $b_m \in \mathbb{C}$ are arbitrary and fixed independently of every variable row.

Proposition 11 (The cube-factor block norm)**.** *For every $\varepsilon > 0$, the raw Gram on this actual block obeys*

$$\sum_{u \text{ in the block}} \left| \sum_m b_m \chi_m(u) \right|^2 \ll_{\nu,S,\varepsilon} (HD)^{\varepsilon} \left\{ R_0 + \frac{DR_0}{A_3} \right\} \sum_m |b_m|^2.$$

The constant is uniform in the fixed unit/S-factor, frozen good factors, H, D and the coefficient sequence. There is no rational-norm-squarefree restriction. Consequently, with $M = \max(A_1, A_2)$, the old powered-BGL bound and this bound give

$$\Lambda_A \ll_{\nu,S,\varepsilon} (HD)^{\varepsilon} \left\{ R_0 + \min \left(\frac{DR_0}{M} + \frac{D^{2/3} R_0}{M^{1/3}}, \frac{DR_0}{A_3} \right) \right\}. \tag{16}$$

The external analytic input is Goldmakher–Louvel, arXiv:1112.1642v2, Corollary 1.2 with $n = 6$, already an explicit classical cube-power sieve. The refinement is this application, not a new large-sieve theorem. For the original normalized coefficients $b_m = D^{-1/2} A_U(m) \nu(m) W(Nm/D)$, the accepted divisor bound gives $\sum_m |b_m|^2 \ll_\eta \|W\|_\infty^2 D^\eta$, uniformly in $U \geq 1$ with $U^2 \leq D$ (and the common W may be complex). Choosing smaller internal exponents yields the corresponding uniform block energy envelope. General cutoff or factor-count weights remain inside the arbitrary coefficient norm; that norm is not silently replaced by one. Summing S-supported row factors uses the reduced capacity $H/N\mathfrak{b}$ and the established convergent geometric sums.

On the complete old saturated face $A_1 = A_2 = A$, $A_3 = B$, $A_4 = A_5 = A_6 = 1$, $AB = H^{1/3}$, the new bound is $A^2 B + DA^2$ in place of the old middle monomial DAB. For $H = D^{3/4}$ put $A = D^a$, $B = D^{1/4-a}$, $0 \leq a \leq 1/4$. When $a < 1/8$, its leading exponent is $1 + 2a < 5/4$; the gain requires B/A to grow by a power for a power improvement. For $a \geq 1/8$, the combined bound still has leading exponent $5/4$. Thus the A_3-dominant part improves, while no new global exponent, BT_σ estimate, signed covariance theorem, Lean theorem or GRH conclusion follows.

Inspecting the surviving sector. The quadratic-pivot inspector plots the complete face, the five monomial powers and the remaining target gap. These are analytic bound exponents, not measured operator norms or trained-network predictions. The refined norm can be inserted in the earlier covariance budget; it does not remove its within-level energies.

A necessary near-face condition. Put $\mathcal{M}_A = DR_0 / \max(M, A_3)$. The combined envelope is at most $(HD)^{\varepsilon}\{R_0 + \mathcal{M}_A + D^{2/3} R_0 / M^{1/3}\}$. If $\mathcal{M}_A \geq DH^{1/3} D^{-\eta}$, then the earlier near-face slack is at most $3\eta \log D$ and, additionally, $A_3 \leq MD^\eta$. Thus a uniform remaining leading-term estimate may focus on this smaller neighborhood; it must still cover the entire $B \leq A$ part of the face. This is a necessary condition on explicit upper-bound monomials, not a lower bound for actual arithmetic energy.

13 Joint completion before conductor absolute values

In the original C–C expansion write $m = hr, n = ht$ with h, r, t pairwise coprime squarefree, $q = rt \neq 1$, $L = hq$, $Q = Nq$, $\ell = NL$. Set $\theta = x_r \overline{x_t}$ and $\xi_q = x_q^3$. The original internal masks give $\theta^3 = \xi_q$ and primitive θ^j on q for $1 \leq j \leq 5$, while $\mu(hr)\mu(ht) = \mu(q)$. Fix generators α of h, β of q, choosing $\alpha = 1$ for $h = 1$. With $\delta = 1 + 2\omega$, put $e_\beta(z) = \exp(2\pi i \operatorname{Tr}(z/(\delta\beta)))$, $\tau_\beta(\eta) = \sum_{z \bmod q} \eta(z) e_\beta(z)$, and $c_h(k) = \prod_{p|h} (Np \mathbf{1}_{k \in p} - 1)$, $c_1 = 1$. For $f_j(v) = \theta(v)^j \mathbf{1}_{(v,h)=1}$,

$1 \le j \le 5$, $f_6(v) = \mathbf{1}_{(v,L)=1}$ and $G_j(k) = \sum_{v \bmod L} f_j(v) e_{\alpha\beta}(kv)$, joint completion gives

$$\prod_{j=1}^{6} G_j(k_j) = Q^2 \xi_q(-\alpha)\tau_\beta(\xi_q)\theta\Big(\overline{\prod_{j=1}^{5} k_j^j}\Big) \prod_{j=1}^{5} c_h(k_j) c_L(k_6). \tag{17}$$

This holds when the first five frequencies are q-units; otherwise the left side is zero. The mixed Gauss phase is orientation-independent, with every common-prime zero retained. The exact row Fourier weight still contains all squarefree, ray, inter-factor, capacity and profile restrictions, including the full A_3 face. Its weighted joint conductor/frequency estimate remains open. This identity and its finite Parseval budget give no improved global exponent or GRH conclusion.

14 A radical-conductor refinement of the raw envelope

Retain the original literal symbols $\chi_m(u) = (u/m)_6$, extended by zero, the fixed good set S, and the squarefree ideal columns $D \le Nm \le 2D$. On an actual dyadic row block write

$$u = u_0 r_1 r_2^2 r_3^3 r_4^4 r_5^5 r_6^6, \qquad A_j \le N r_j < 2A_j, \qquad \prod_{j=1}^{6} A_j^j \le H.$$

Only $r_1, \dots, r_5$ are pairwise-coprime squarefree ideals; r_6 is arbitrary and may overlap them. Fix primary generators, a literal unit and an S-supported factor in u_0. Put $R = \prod_j A_j$, $M = \max(A_1, A_2)$ and $Q = R/A_6$. For arbitrary complex coefficients b_m, independent of every variable row, define

$$\mathcal{E}_A(b) = \sum_{u \text{ in the actual block}} \Big| \sum_m b_m \chi_m(u) \Big|^2, \qquad \Lambda_A = \sup_{b \ne 0} \frac{\mathcal{E}_A(b)}{\sum_m |b_m|^2}.$$

Proposition 12 (An ordinary conductor comparison)**.** *For every $\varepsilon > 0$,*

$$\Lambda_A \ll_{S,\varepsilon} (HD)^\varepsilon A_6 \{D + Q^2\}. \tag{18}$$

Consequently the original residue-subtracted remainder satisfies, uniformly for $H, D, U \ge 1$ and $U^2 \le D$,

$$\sum_{0 < Nu \le H} |E_u|^2 \ll_{\nu,S,\varepsilon} \|W\|_\infty^2 (HD)^\varepsilon \{H + DH^{1/6} + (DH)^{2/3}\}. \tag{19}$$

Here ν, S are fixed and $W \in C_c^\infty((0,\infty))$ is one common, possibly complex profile supported in $[1,2]$; there is no profile supremum.

The input to (18) is the classical ordinary sieve for *all* primitive characters of conductor norm at most Q, with constant $D + Q^2$, not a higher-order-family theorem for arbitrary conductor labels. The following section gives its standard Gauss-sum and two-dimensional lattice proof, including prime-power conductors at S. For fixed r_6, the first five factors have primitive outside-S conductor $q = r_1 \cdots r_5$. The literal primitive character agrees with the raw symbol even at collisions with q; the sixth-power factor instead remains the coefficient mask $\mathbf{1}_{(m,r_6)=1}$. At most $5^{\omega(q)} \ll_\eta (Nq)^\eta$ tuples give the same primitive character. Applying the ordinary sieve and then counting r_6 proves (18). No rational-norm-squarefree restriction is imposed.

To obtain (19), combine this with the accepted powered-BGL block bound from Section 8. With $S_A = R/M$ and $T_A = D^{2/3}R/M^{1/3}$, capacity gives $R \le H$, $A_6 \le H^{1/6}$, $RS_A \le H$ and $T_A \le (DH)^{2/3}$. Hence

$$\begin{aligned} \min(R + DS_A + T_A, DA_6 + R^2/A_6) &\le R + T_A + DA_6 + \min(DS_A, H^2/S_A^2) \\ &\le H + DH^{1/6} + 2(DH)^{2/3}. \end{aligned}$$

Dyadic and fixed-S summation, the uniform actual coefficient norm, and the principal-residue estimate give the assertion for $E_u = I_u - P_u$.

At $H = D^{3/4}$ the new envelope has exponent $7/6$, improving the old $5/4$ by $1/12$; its excess above the target H is still $D^{5/12}$. On the *entire* old face $A_1 = A_2 = A$, $A_3 = B$, $A_4 = A_5 = A_6 = 1$, $AB = H^{1/3}$, one has $Q = A^2B = AH^{1/3} \le H^{2/3}$, so (18) costs $D^{1+\varepsilon}$ there. This includes $B \le A$. The new combined upper bounds reach $7/6$ at $(\log_D A_1, \log_D A_2) = (5/12, 1/6)$ with other coordinates zero; the pure point $\log_D A_1 = 3/4$ also retains the old BGL cross term. These are upper-envelope witnesses, not arithmetic energy lower bounds or counterexamples to signed cancellation.

The simpler proposed replacement of the radicand ab^2 by its radical ab inside BGL remains unjustified: after the existing fixed reciprocity phases, the mixed reciprocity quotient is $\vartheta_b(a')/\vartheta_a(b')$, which varies outside S. The present application does not require that replacement. No target signed moment, type-II covariance estimate, positive arithmetic determinant comparison, zero-location result, new Lean theorem or GRH conclusion follows from (19).

15 Learning the actual signed arithmetic correlations

The unsigned refinement in Section 14 leaves a signed problem. Here is a separate, executed finite learning objective on the original correlations, with a certificate for every coordinate and observable. It is not a uniform estimate for an expanding family. For ordered distinct squarefree columns $m = hr, n = ht$, put $q = rt$ and

$$\Gamma_i = \sum_{u \in \mathcal{R}} \chi_r(u)\overline{\chi_t(u)}\mathbf{1}_{(u,h)=1}, \qquad i = (h, q, r).$$

The symbols vanish at every collision with q; the common-prime mask is retained separately. With principal ν, actual complementary divisor counts $r_U(m)$, and one common complex profile $W_m = W(Nm/D)$, define

$$d_i^{CC} = r_U(m)r_U(n)W_m\overline{W_n}, \qquad d_i^{C\mu} = r_U(m)W_m\overline{W_n}, \qquad \lambda_i = 1 + |d_i^{CC}|^2 + |d_i^{C\mu}|^2.$$

The exact diagonal Δ_X and these offdiagonal coordinates give

$$E_X = \Delta_X + D^{-1}\sum_i \mu(q_i)d_i^X\Gamma_i, \qquad X \in \{CC, C\mu\}. \tag{20}$$

Thus $E_{CC} = \sum_{u \in \mathcal{R}} |C_u|^2$ and $E_{C\mu} = \sum_{u \in \mathcal{R}} C_u\overline{M_u}$ for the actual normalized row sums. Neither the Möbius sign nor a complex conjugate is discarded.

Proposition 13 (A finite supervised certificate)**.** *Let g_i be complex readout coordinates and $L(g) = \sum_i \lambda_i|g_i - \Gamma_i|^2$. For $K_X = \sum_i |d_i^X|^2/\lambda_i$ and $\lambda_{\min} = \min_i \lambda_i$, the substituted readouts satisfy*

$$|\widehat{E}_X - E_X|^2 \le D^{-2}K_X L(g),$$
$$\max_i |g_i - \Gamma_i|^2 \le L(g)/\lambda_{\min}.$$

The executed metric-gradient update $g^{(t+1)} = (g^{(t)} + \Gamma)/2$ gives $L(g^{(t)}) = 4^{-t}L(g^{(0)})$. Therefore every positive rational requested tolerance on these three squared bounds is attained after finitely many updates on this fixed exact instance, with unrestricted resources.

The implementation's resource guard refuses requests beyond its update budget; termination in this proposition is not a practical runtime guarantee. The labels Γ_i are computed exactly before training, so no speedup over exact label evaluation or generalization to unseen conductors is claimed.

The supplied instance uses $D = 6000, U = 60, H = 625$, S above $2, 3$, principal ν, and all 15 good squarefree ideal columns at activated norms $7507, 7519, 8281, 10507$ (respectively $2, 4, 1, 8$ ideals). One explicitly specified common smooth four-bump profile is zero at every other integer norm. Both the 2262 literal rows and the separately labelled 2148 guaranteed nonprincipal subset are evaluated; the latter is not asserted to be the whole nonprincipal family. There are 95 offdiagonal (h, q) blocks, 65 with $h = 1$, and 210 ordered learned coordinates. Every row case retains 4530 raw column zeros. Real and complex profiles give four requested-10^{-12} certificates, using $53, 55, 53, 55$ metric updates respectively: 216 executed updates, with every full rational state saved. A separate 200 executed ordinary coordinate updates are budget-limited and carry no requested-error claim. The standalone replay is linked from the book and project page. No analytic saving, positive determinant, zero-location assertion, upstream quantum training, new Lean declaration or GRH conclusion follows.

16 Why the common profile must stay fixed

The unsigned envelope of Section 14 and the unresolved signed estimate have different profile quantifiers. We give an obstruction to an H-sized bound uniform through the sup norm alone. It uses the *actual* coefficients and nonprincipal rows, rather than replacing the columns by arbitrary weights. It does not contradict an estimate for one fixed smooth profile or one with adequate derivative dependence.

Fix $K = \mathbb{Q}(\omega)$, $\omega^2 + \omega + 1 = 0$, principal $\nu = 1$, and a finite good set S containing the primes above $2, 3$. Write $\zeta = 1 + \omega$, $\delta = 1 + 2\omega$, so ζ has order six and $\delta^2 = -3$. Retain the literal raw symbols and normalization

$$I_u(W) = D^{-1/2} \sum_{m \text{ squarefree } (m,S)=1} A_U(m)\chi_m(u)W(Nm/D), \qquad E_u = I_u - P_u.$$

Here and below squarefree means *ideal* squarefree. For $U^2 = D$ the accepted high-window identity gives $A_U(m) = \mu(m)(r_U(m) - 1)$ on $D < Nm < 2D$, with $r_U(m)$ even and $|A_U(m)| \geq 1$. The multiset of prime-ideal norms of a squarefree ideal is determined by its rational norm. Thus $A_U(m) = A_N$ depends only on $N = Nm$, including divisor-norm multiplicities. Moreover

$$\eta(m) := \chi_m(\zeta) = \zeta^{(Nm-1)/6}, \qquad (m, S) = 1. \tag{21}$$

The primitive character η has conductor $(2)^2(\delta)^3$, of norm 432; its powers $1, 5$ have conductor norm 432, powers $2, 4$ norm 27, and power 3 norm 16. All five powers are nonprincipal. These row conductors do not remove the raw zero at any prime dividing a sixth factor.

Put $\phi(t) = \exp(1 - 1/(1 - t^2))$ for $|t| < 1$ and zero otherwise. For integer D set $w_N = \operatorname{sgn}(A_N)\overline{\zeta^{(N-1)/6}}$ when a good squarefree ideal of norm N exists and $D < N < 2D$, and zero otherwise. Define one common, complex profile

$$W_D(y) = \sum_{D<N<2D} w_N \phi(4(Dy - N)). \tag{22}$$

Its disjoint supports lie strictly inside $(1, 2)$, its sup norm is one for large D, and $\|W_D^{(j)}\|_\infty \ll_j D^j$. In particular the endpoints have value zero. This is a profile varying with scale, independent of the row, not a separate choice for each ideal or each row.

Proposition 14 (A moving-profile lower bound)**.** *For every fixed* $0 < \theta < 1$*, take* $H = D^\theta$*,* $U = \sqrt{D}$*, and* $X = H^{1/6}$*. As integer* $D \to \infty$*,*

$$\sum_{0<Nu\le H} |E_u(W_D)|^2 \ge (c_S + o(1))DH^{1/6}, \qquad c_S > 0. \tag{23}$$

This lower bound already comes from literal rows $u_v = \zeta\beta_v^6$*, where* v *is an arbitrary good ideal of norm at most* X *and* β_v *is a generator. These rows have nonprincipal primitive character* η*, so* $P_{u_v} = 0$*. Their exact values are*

$$E_{u_v}(W_D) = D^{-1/2} \sum_{\substack{m \text{ squarefree, } (m,Sv)=1 \\ D<Nm<2D}} |A_U(m)|. \tag{24}$$

Every collision with v *remains deleted, even when the primitive conductor drops to* 432*. No squarefree condition is imposed on* v*.*

At $H = D^{3/4}$ the lower exponent is $9/8$, whereas the current upper exponent is $7/6$. Thus this does not establish sharpness of $7/6$ or a new upper saving. For $0 < \theta < 2/3$, the term $DH^{1/6}$ dominates the current unsigned envelope; it is necessary up to subpower losses in this *varying-profile* class, even on nonprincipal rows.

For $k \ge 0$ put $M_k(W) = \sum_{j=0}^k \|(y\partial_y)^j W\|_\infty$. If, for every $\varepsilon > 0$, a bound uniform over all common profiles with constants independent of W, D were to assert

$$\sum_{0<Nu\le H} |E_u(W)|^2 \ll_\varepsilon M_k(W)^\alpha (HD)^\varepsilon H, \qquad \alpha \ge 0,$$

then necessarily $k\alpha \ge 1 - 5\theta/6$. At $\theta = 3/4$ this is $k\alpha \ge 3/8$. For a single fixed ε the necessary condition is only $k\alpha + (1+\theta)\varepsilon \ge 1 - 5\theta/6$. These are necessary cost comparisons, not sufficient estimates. A fixed-W result, a class with bounded seminorms, or the existing $M_4(W)^2$ dependence is not refuted.

The same rows expose a concrete centering error in the joint completion: removing the orientation mean of the actual correlation can delete a nonzero signed contribution on *nonprincipal* rows. The extended proof retains the literal unit phase, gives an exact four-orientation example, and explains why orientation projection alone supplies no saving. The standalone profile and coherent-mean inspector shows the complete finite norm groups, arbitrary sixth-factor masks and exact residual quantities. Its finite tables are diagnostics of this construction, not asymptotic estimates, training or GRH evidence.

17 A bound for the signed coprime slice and an exact character twist

The coprime-column contribution can be negative, so it needs its own bound. Fix the original ν, S, one common possibly complex smooth W supported in $[1,2]$, $D > 1$, $U \ge 1$ and $U^2 \le D$. Consider an actual block

$$u = u_0 v_1 v_2^2, \quad A_j \le Nv_j < 2A_j \ (j = 1,2), \quad Nu \le H, \qquad R = A_1A_2, \quad M = \max(A_1, A_2).$$

The good ideals v_1, v_2 are squarefree and coprime, with their literal primary/ray representatives; all other good factors are the unit ideal. Let F be the actual indicator, including capacity and every

row mask. In the remaining neighborhoods the first factor is nonunit, so the primitive row character is nonprincipal and $E_u = I_u$ there. For $X, Y \in \{A_U, C_U, \mu\}$ define the actual signed slice

$$S_1^{XY} = \frac{1}{D} \sum_u F(u) \sum_{\substack{m,n \text{ squarefree,good} \\ (m,n)=1}} X(m)Y(n)\nu(m)\overline{\nu(n)}\chi_m(u)\overline{\chi_n(u)}W(Nm/D)\overline{W(Nn/D)}. \quad (25)$$

All columns in this formula are the original annular columns, including raw zero sentinels. No positive-semidefinite claim about the coprime truncation is made.

Proposition 15 (A bound for the actual signed $h = 1$ slice)**.** *Uniformly in the original cutoff and this actual block,*

$$|S_1^{XY}| \ll_{\nu,S,\epsilon} \|W\|_\infty^2 (HD)^\epsilon \min\left\{R + \frac{DR}{M} + \frac{D^{2/3}R}{M^{1/3}},\ D + R^2\right\}. \quad (26)$$

The exact identity underlying the estimate is

$$\begin{aligned} DS_1^{XY} &= \sum_{\substack{d \text{ squarefree} \\ (d,S)=1}} \mu(d) \sum_{\substack{u \text{ in the actual block} \\ (d,u)=1}} T_d^X(u)\overline{T_d^Y(u)}, \\ T_d^X(u) &= \sum_{\substack{a \text{ squarefree} \\ (a,dS)=1}} X(da)\nu(a)\chi_a(u)W(Nd\,Na/D). \end{aligned} \quad (27)$$

In particular $X(da)$ is not replaced by $X(d)X(a)$, and the mask $(d, u) = 1$ remains.

The proof applies the accepted block sieves inside each positive row norm in (27), then explicitly loses the d signs in an absolute triangle. At $H = D^{3/4}$ the two remaining power-scale neighborhoods $A_1 \asymp D^{5/12}, A_2 \asymp D^{1/6}$ and $A_1 \asymp D^{3/4}, A_2 \asymp 1$, with feasible capacity constants, both retain power $7/6$. The desired $3/4$ still needs a signed saving of $5/12$. This proves a separate signed-slice bound; it does not improve the published global envelope or supply the missing cancellation.

There is also an exact arithmetic description of the two-pivot phase. Put $\delta = 1 + 2\omega$ and, at each good prime ideal p, set $j(p) = J_p(\chi_p, \chi_p^2)$. Extend j multiplicatively on squarefree columns. Let $\Psi(a)$ be the unique generator of an ideal prime to 2δ satisfying $\Psi(a) \equiv 1 \pmod{2\delta}$.

Proposition 16 (The actual Jacobi column twist)**.** *The map Ψ is an algebraic Hecke character with exact finite conductor (2δ), of norm 12, and principal convention $\Psi((\alpha)) = \alpha$ for $\alpha \equiv 1 \pmod{2\delta}$ (infinity type $(1, 0)$). For good squarefree columns,*

$$(j(p)) = p, \quad -j(p) = \Psi(p), \qquad \mu(m)j(m) = \Psi(m). \quad (28)$$

Define $\vartheta(m) = \Psi(m)/\sqrt{Nm}$ and $\rho(m) = j(m)/\sqrt{Nm} = \mu(m)\vartheta(m)$ on those columns. In the high window the actual gauged coefficients are $b_A = (r_U - 1)\vartheta\nu W$, $b_C = r_U\vartheta\nu W$ and $b_\mu = \vartheta\nu W$.

This does not replace the original row kernel. Its compensating inverse gauge remains, so all ordinary coefficient/operator norms are unchanged. The new character has nonzero infinity type and is not silently covered by a theorem stated only for finite-order twists. The identities in (28) involving μ must not be extended to arbitrary nonsquarefree sixth factors.

A fresh fixed-profile example takes $\nu = 1$, S the primes above $2, 3$, literal unit one and primary row generators 1 modulo 3. It lies in an actual $A_6 = 1$ block at $D = 60000$, $U = 244$, $H = 3833$, $A_1 = 37$, $A_2 = 7$. All 20 rows of the stated literal class are nonprincipal. A selected q has exactly two

legal column orientations, of norms 97051/96091 and its reversal. Their correlations in the ω basis are $(-4,-2)$ and $(-2,2)$, with mean -3. Their selected contribution is $S_{1,q}^{AA} = -1/10000$; deleting the mean gives zero. The centered squared norm is 6, not zero, and the proposed variance-only Cauchy bound would incorrectly require $36 \le 12$. This refutes that centering rule in a fixed-profile row block, not the asymptotic target. Other q contributions remain. The coprime-kernel inspector shows the actual rows, raw zeros, two-column Gram matrix and compensated gauge; no training or arithmetic spectrum is inferred from that finite view.

18 The signed divisor mean and a complete-column laboratory

The preceding signed-slice estimate loses the sign of the common ideal divisor. We now retain that variable in an exact analytic reduction and in a complete finite computation. Neither supplies the missing 5/12 saving at $H = D^{3/4}$. The computations below use a new fixed profile; they do not fill in the uncomputed columns of the earlier broad-plateau profile.

Let $\mathcal{C}$ be original squarefree good ideal columns in $[D, 2D]$, and let $\mathcal{R}$ be an actual finite literal row block, with nonnegative multiplicities w_u. All shared-prime zeros of the original $\chi_m(u)$ remain. For arbitrary complex coefficient vectors define the linear-first form

$$DQ_1(x,y) = \sum_{u\in\mathcal{R}} w_u \sum_{\substack{m,n\in\mathcal{C}\\(m,n)=1}} x_m \overline{y_n}\chi_m(u)\overline{\chi_n(u)}. \tag{29}$$

The original component is obtained with $x_m = X_U(m)\nu(m)W(Nm/D)$ and the corresponding y_m. On the nonprincipal row blocks used below the original residue term vanishes. Equation (29) is a raw coprime form on a generic block; principal rows would still require their separate residue assembly.

For every ideal divisor of an active column put

$$F_d(x;u) = \sum_{m\in\mathcal{C},\ d\mid m} x_m\chi_m(u), \qquad Z_d(x,y) = \sum_u w_u F_d(x;u)\overline{F_d(y;u)}.$$

The exact identity is

$$DQ_1(x,y) = \sum_d \mu(d) Z_d(x,y). \tag{30}$$

There is no extra $\mu(q)$: the vectors already contain the original coefficients. Full divisors $d = m$ are essential to cancel self-pairs. Divisors and squarefreeness refer to ideals, including distinct ideals with the same rational norm.

Proposition 17 (A weighted divisor-incidence lift)**.** *Suppose $\lambda_d \ge 0$ is an arbitrary-coefficient bound*

$$\sum_u w_u |F_d(x;u)|^2 \le \lambda_d \sum_{m\in\mathcal{C},\ d\mid m} |x_m|^2.$$

With $B(m) = \sum_{d\mid m}\lambda_d$ and $\|x\|_B^2 = \sum_m B(m)|x_m|^2$, one has

$$|DQ_1(x,y)| \le \|x\|_B \|y\|_B. \tag{31}$$

For a finite literal phase table one may take the exactly computed

$$\lambda_d = E_d := \sum_u w_u \#\{m\in\mathcal{C} : d \mid m,\ \chi_m(u)\ne 0\}.$$

This last choice is a Frobenius certificate, not a new analytic estimate. For $x = t + e$ it gives the finite perturbation bound

$$|Q_1(x,x) - Q_1(t,t)| \le D^{-1}(2\|t\|_B\|e\|_B + \|e\|_B^2). \tag{32}$$

Proof. Insert ideal-divisor inversion in (29). Apply row Cauchy to each Z_d and weighted Cauchy to the resulting sum over d. Then $\sum_d \lambda_d\|x_d\|_2^2 = \|x\|_B^2$. The squared Frobenius norm of the actual F_d matrix is exactly E_d. Expand $Q_1(t+e,t+e) - Q_1(t,t)$ to obtain the last assertion. Positivity of the coprime-truncated matrix was never assumed. □

Using an accepted positive-row sieve bound for λ_d only recovers the old exponent up to ideal-divisor factors. It does not estimate the signed sum before the triangle in (30).

There is also a genuinely signed scalar expansion. Fix squarefree residual ideals a, b, which may overlap, and an actual row u. Keep the full deleted set $\mathcal{T} = S \cup \operatorname{supp}(abu)$ and set

$$\mathcal{Z}_{U;a,b,\mathcal{T}}(s) = \sum_{\substack{d \text{ squarefree} \\ (d,\mathcal{T})=1}} \frac{\mu(d)A_U(da)A_U(db)}{(Nd)^s}, \qquad \Re s > 1.$$

For four positive cutoff variables put $P_p = 1-(Np)^{-v_1}-(Np)^{-v_2}$ and $Q_p = 1-(Np)^{-w_1}-(Np)^{-w_2}$. The common-divisor Euler factor is $1-(Np)^{-s}P_pQ_p$. Its first-order factorization has four numerator and five denominator zeta factors:

$$\prod_{p\notin\mathcal{T}}(1-(Np)^{-s}P_pQ_p) = \frac{\prod_{i=1}^2 \zeta_K^{\mathcal{T}}(s+v_i)\zeta_K^{\mathcal{T}}(s+w_i)}{\zeta_K^{\mathcal{T}}(s)\prod_{i,j=1}^2 \zeta_K^{\mathcal{T}}(s+v_i+w_j)}\,\mathcal{H}_{\mathcal{T}}(s;v,w). \tag{33}$$

The correction is normally convergent at $\Re s \ge \frac12+\eta$, $\Re v_i, \Re w_j \ge 0$, for each fixed $\eta > 0$. This says nothing about zeros or residues of the zeta ratio. Finite-height Perron kernels, with their explicit endpoint/error costs, recover the cutoffs by successive limits.

A direct finite cutoff expansion gives more explicit information:

$$\mathcal{Z}_{U;a,b,\mathcal{T}}(s) = \frac{\mathcal{N}_{U;a,b,\mathcal{T}}(s)}{\zeta_K^{\mathcal{T}}(s)}. \tag{34}$$

The numerator is a finite sum over eight nonzero joint short-divisor states, holomorphic on $\Re s > 0$. Uniformly for $\sigma \ge \sigma_0 > 0$,

$$|\mathcal{N}_{U;a,b,\mathcal{T}}(\sigma+i\tau)| \ll_{K,\sigma_0,\epsilon} (Na\,Nb)^{\epsilon}U^{4\max(1-\sigma,0)+\epsilon}.$$

The costs near $\sigma = 1/2$ persist. The remaining long state has the Möbius sign, but the eight finite short states cannot be omitted. A fixed-residual reciprocal zeta factor also does not prove an uncancelled pole in the complete coherent sum.

Exact complete-column reference. Choose once $\delta = 1/240000$ and $W_\delta(y) = s((y-1)/\delta)s((2-y)/\delta)$, where s is the smooth step from zero to one. For the finite grid $D \le 60000$ every strict integer norm node has $W_\delta = 1$; endpoints vanish. Its derivatives cost $O_k(\delta^{-k})$. The three cases have different declared row blocks, not a common asymptotic scaling experiment. They use principal ν, S above $2, 3$, literal class one and nonprincipal actual rows.

D	columns	divisors	rows	S_1^{AA}
144	41	53	5	$-19/36$
576	164	212	11	$-221/288$
60000	16953	21750	20	$-5119/5000$

In the last case the exact even/odd Möbius masses are 543416 and 604844, giving numerator -61428 before division by D. All 339060 raw column–row values, including 47908 zeros, and all nine cross components are retained. In particular 1361 columns have nonsquarefree rational norm while remaining squarefree as ideals. The signed-divisor inspector exposes the actual states and dimensions. These finite negative slices are neither whole-family asymptotic counterexamples nor a positive arithmetic determinant or GRH result.

19 A target-sized budget for rough-prime masks

The actual signed divisor mean admits a useful quantitative simplification: the rough-prime coprimality mask can be restored with an explicit correction bound. The remaining main term still retains the Möbius sign and the coherent row mean; it is not estimated by this result.

Work over $\mathbb{Z}[\omega]$. Fix a finite S containing the primes above $2, 3$, a unitary multiplicative ν on good ideals, and one common complex profile W supported on $[1, 2]$, smooth and zero at both endpoints. Put $B = \|W\|_\infty$ and assume

$$1 \le U, \qquad U^2 \le D, \qquad 2D \le 7U^2, \qquad H \ge 1. \tag{35}$$

Let $\mathcal{R}$ be the actual finite family of nonzero literal rows, with original weights $w_u \ge 0$, $1 \le Nu \le H$, and $R_w = \sum_u w_u$. All original internal squarefree/coprime conditions, unit and S-ray classes, capacity and sixth-factor masks remain in this family. Write S_1^{AA} for the raw form (29) with $x_m = A_U(m)\nu(m)W(Nm/D)$, where

$$A_U(g) = \sum_{\substack{xy \mid g,\ (x,y)=1 \\ Nx \le U,\ Ny \le U}} \mu(x)\mu(y). \tag{36}$$

Here the columns in (29) are the COMPLETE original good squarefree annulus $[D, 2D]$, with the common W as its norm cutoff and the original full-good-ideal/ν, S formulation. No selected-column indicator or additional column-ray filter is allowed in this reduction; all actual row unit/ray/capacity filters remain. Principal-row residues, if present, require their separate assembly. For the intended $A_1 > 1$ blocks the defining conductor of ν is supported in S, and the rows are nonprincipal.

Let f range over good squarefree U-smooth ideals, put $F = Nf$, and set

$$T_f(t; u) = \sum_{\substack{a \text{ good squarefree} \\ (a,f)=1}} A_U(fa)\nu(a)\chi_a(u)W(FtNa/D). \tag{37}$$

Here $\chi_a(u) = 0$ on $(a, u) > 1$. For a prime ideal p, let $T_f^{(p)}(t; u)$ impose the additional condition $(a, p) = 1$. The smooth factor f is distinct from the original column-pair gcd sector $h = 1$. Define the following *unnormalized* expressions:

$$K_0 = \sum_u w_u \sum_{\substack{f \text{ good SF, } U\text{-smooth} \\ (f,u)=1}} \mu(f)|T_f(1; u)|^2, \tag{38}$$

$$K_T = \sum_u w_u \sum_{\substack{f \text{ good SF, } U\text{-smooth} \\ (f,u)=1}} \mu(f) \sum_{\substack{p \notin S,\ Np > U \\ p \nmid u}} |T_f^{(p)}(Np; u)|^2, \tag{39}$$

$$K_S = \sum_u w_u \sum_{\substack{f \text{ good SF, } U\text{-smooth} \\ (f,u)=1}} \mu(f) \sum_{p \notin S,\ Np > U} |T_f(Np; u)|^2. \tag{40}$$

Throughout, p denotes a prime ideal. None of these expressions is declared positive, since the $\mu(f)$ signs remain.

The exact decomposition of the common ideal divisor $d = fe$ has at most one rough prime when the residual pair is nonunit. The unit residual pair also permits two distinct rough primes, necessarily with $f = 1$. Its exact correction is

$$C_2 = \sum_{\substack{\{p,q\} \text{ unordered, } p\neq q \\ p,q\notin S,\ Np,Nq>U}} |W(NpNq/D)|^2 \sum_{\substack{u\in\mathcal{R} \\ (u,pq)=1}} w_u. \tag{41}$$

Conjugate primes of equal rational norm are included. One has

$$DS_1^{AA} = K_0 - K_T + C_2. \tag{42}$$

In K_S, terms with $p \mid a$ or $p \mid u$ are explicitly auxiliary: fpa can fail to be squarefree. They use $A_U(fa)$ as written, not a new original column or a substitution for $A_U(fpa)$.

Theorem 18 (Explicit rough-prime mask budget)**.** *Put $\kappa_U(u) = \#\{p \notin S : p \mid u,\ Np > U\}$ and $\Delta = (K_S - K_T)/D$. Under* (35)*,*

$$|\Delta| \le B^2 \left(229376 \sum_u w_u \kappa_U(u) + 2048 R_w \right), \tag{43}$$

$$0 \le C_2/D \le 1792 B^2 R_w. \tag{44}$$

Consequently

$$S_1^{AA} = \frac{K_0 - K_S}{D} + E, \tag{45}$$

$$|E| \le B^2 \left(229376 \sum_u w_u \kappa_U(u) + 3840 R_w \right). \tag{46}$$

If $H \le D^{3/4}$, then every row has $\kappa_U(u) \le 1$, and in particular

$$|E| \le 233216 B^2 R_w \le 2^{18} B^2 R_w. \tag{47}$$

Only the rough-prime deletion has been restored; the smooth-divisor mask, literal row family and original coefficients remain. The explicit bound uses the exact restricted identity

$$A_U(g) = \mu(g)(2\mathbf{1}_{Ng\le U} - 1) \quad (Ng \le 7U,\ g \text{ good squarefree}).$$

Every active term of $T_f(Np; u)$ has $N(fa) < 7U$, hence $|A_U(fa)| = 1$ there. This is not a shrinkage or proxy for the whole original coefficient family. Only absolute correction bounds use it; the signed coefficients in the clean main remain unchanged.

For general H, one has $\kappa_U(u) \le 2\log(2H)/\log(2U)$, also at $U = 1$. For an actual two-pivot block with $R_w \ll A_1A_2 \le H$ and $H \le D^{3/4}$, (47) gives an $O(B^2H)$ error with no divisor-power or profile-derivative loss. The remaining target problem for this raw A–A slice is the clean signed main $(K_0 - K_S)/D$. It has not been bounded here; the 7/6 envelope and missing 5/12 improvement for the complete original remainder are unchanged. The argument neither deletes a coherent mean nor treats the coprime-truncated form as positive.

Three exact finite controls. These use principal ν, S above $2,3$, literal unit class one and complete original good squarefree annuli. The first two row variables are squarefree, good and coprime and may include the unit ideal; all other factors equal one.

D	U	H	A_1	A_2	columns	rows
200	10	53	13	1	52	5
512	16	107	19	1	139	7
900	30	384	7	7	250	2

Put $\eta_0(t) = \exp(-1/t)$ for $t > 0$ and $\eta_0(t) = 0$ for $t \le 0$, and define

$$s(t) = \frac{\eta_0(t)}{\eta_0(t) + \eta_0(1-t)}.$$

Choose once $\delta = 1/4096$ and $W_\delta(t) = s((t-1)/\delta)s((2-t)/\delta)$. For integer $D \le 1024$, strict integer norm nodes have weight one and endpoints have weight zero; its derivatives cost $O_k(4096^k)$. The following numbers are RAW, before division by D:

D	K_0	K_S	K_T	E_{row}	E_{res}	C_2	DS_1^{AA}
200	339	235	209	20	6	21	151
512	366	781	711	48	22	89	−256
900	436	410	366	0	44	28	98

In every case $K_S - K_T = E_{\mathrm{row}} + E_{\mathrm{res}}$ and $DS_1^{AA} = K_0 - K_T + C_2$. The displayed nonnegative values of $E_{\mathrm{row}}, E_{\mathrm{res}}$ are finite observations; no general nonnegativity is assumed for either signed correction. Only C_2 is nonnegative by definition. The first two capacities are $\lfloor D^{3/4} \rfloor$. The third $H = 384$ is a declared noncritical fixture capacity, not $\lfloor 900^{3/4} \rfloor$; both of its rows have $\kappa_U = 0$. All ideal and raw-zero masks, unit residual states and explicitly auxiliary non-squarefree p^2 terms are retained. The exact control summary records the scopes and exact fractions after one division by D. These are finite identities, not a proof or numerical fit of the analytic bounds, and do not estimate the signed clean main.

20 The coherent signed main and its remaining inputs

The physical sixth-power-free comparison in Section 21 provides a new positive upper bound. We now collect the exact coherent reductions that it can enter. They preserve the original mean and all raw masks; none proves the target by a scalar cancellation assertion.

Use the complete original good squarefree annulus and common complex profile of Section 19. Write $\mathcal{U}$ for the actual nonzero literal rows, $w_u \ge 0$, $R_w = \sum_u w_u$ and $B = \|W\|_\infty$. Retain fixed S, ν, all internal row masks and (35). Every identity below is a raw A–A identity; principal residues still require their separate assembly. Classical operator specializations are only for the original unweighted $A_6 = 1$ rows, or their already justified bounded natural multiplicities.

First there is a useful distinction between two auxiliary kernels. Let K_{all} be K_S of (40) with the U-smooth restriction on its squarefree f removed, retaining $(f,u) = 1$ and exactly $A_U(fa)$ in every T_f. Set

$$\begin{aligned} P_2 &= \sum_u w_u \sum_{\substack{p \notin S \\ Np > U}} \sum_{\substack{q \notin S,\ Nq > U \\ q \nmid u}} |W(NpNq/D)|^2, \\ K_S &= K_{\mathrm{all}} + P_2, \qquad 0 \le P_2/D \le 3584 B^2 R_w. \end{aligned} \tag{48}$$

Here (p,q) is ordered, $p=q$ is allowed, and only q has the stated row mask. This is distinct from the original unordered, distinct-prime C_2 in (41). In particular

$$S_1^{AA}=\frac{K_0-K_{\mathrm{all}}}{D}+\Delta+\frac{C_2-P_2}{D}. \tag{49}$$

At $H\le D^{3/4}$ the error in this *new* auxiliary main is at most $236800B^2R_w$. The old $233216B^2R_w$ bound for $(K_0-K_S)/D$ is unchanged. The physical comparison in (57) estimates the new combined main, in its stated $A_6=1$ scope, without estimating its two summands separately.

The prime-discrepancy clauses below additionally require $U\ge 7$. Define

$$E_S(t)=\pi_K^S(t)-\mathrm{Li}_2(t),\qquad \mathrm{Li}_2(t)=\int_2^t\frac{dv}{\log v},\qquad M_1(W)=\max\{B,\|xW'(x)\|_\infty\}.$$

The count is of prime ideals outside fixed S, including distinct conjugate ideals and inert prime norms. Put

$$Q(t)=\sum_{\substack{f\ \text{good SF}\\ f\ U\text{-smooth}}}\mu(f)\sum_{\substack{u\in\mathcal{U}\\(f,u)=1}}w_u|T_f(t;u)|^2,\qquad \mathcal{J}=\int_U^\infty\frac{Q(t)}{\log t}\,dt.$$

Proposition 19 (A joint form retaining its mean)**.** *On the finite space of good squarefree g with $Ng\le 2D/U$, let*

$$b_g(t)=A_U(g)\nu(g)W(Ng\,t/D),\quad (\mathsf{R}c)(u)=\sqrt{w_u}\sum_g c_g\chi_g(u),\quad (\mathsf{P}_f c)_g=\mathbf{1}_{f\mid g}c_g.$$

With ordinary counting ℓ^2 norms,

$$Q(t)=b(t)^*\mathcal{M}_U b(t),\qquad \mathcal{M}_U=\sum_{f\ U\text{-smooth}}\mu(f)\mathsf{P}_f\mathsf{R}^*\mathsf{R}\mathsf{P}_f,\quad t\ge U. \tag{50}$$

The matrix is self-adjoint, with no positivity assertion. The exact prime substitution is

$$K_S=\mathcal{J}+\mathcal{E}_{\mathrm{PIT}},\qquad \mathcal{E}_{\mathrm{PIT}}=-E_S(U)Q(U)-\int_U^\infty E_S(t)Q'(t)\,dt. \tag{51}$$

The endpoint excludes $Np=U$ exactly. It must remain.

For any true arbitrary-coefficient row norm $\lambda_*(L)$, use the direct unit-column value R_w on $1/2\le L<1$. On the active range $1\le L\le D/U$, the current original two-pivot bound is the minimum in Proposition 25; it is not a theorem for arbitrary row weights. Let $\lambda^{\#}(D/T)$ dominate this norm on the two adjacent active length dyads. Then

$$\frac{|\mathcal{E}_{\mathrm{PIT}}|}{D}\ll_\epsilon (HD)^\epsilon M_1(W)^2\left\{\frac{|E_S(U)|}{U}\lambda_*(D/U)+\sum_T\mathfrak{d}_2(T)\lambda^{\#}(D/T)\right\}, \tag{52}$$

where $T=2^jU<2D$ and $\mathfrak{d}_2(T)^2=\int_T^{2T}|E_S(t)|^2t^{-3}\,dt$. This is a sufficient-input budget, not an asserted power estimate for $\mathfrak{d}_2$ or for the endpoint. The pointwise version retains $\lambda_*(D/U)+\int_{1/2}^{D/U}\lambda_*(L)\,dL/L$, with the current minimum inside that integral.

On good rough-prime columns the exact quadratic-form compression is $\Pi_{\mathcal{P}}\mathcal{M}_U\Pi_{\mathcal{P}}=\Pi_{\mathcal{P}}\mathsf{R}^*\mathsf{R}\Pi_{\mathcal{P}}$. This is not an invariant-subspace assertion: off-block couplings remain and can cancel its contribution. Radial Mellin Parseval likewise retains equal-norm ideal correlations, including their derivatives; it neither orthogonalizes orientations nor permits deleting their mean. Even exact prime density leaves $K_0-\mathcal{J}$ to estimate.

Proposition 20 (A sufficient smooth-divisor tail cutoff)**.** *For the original unweighted $A_6 = 1$ two-pivot rows put $R = A_1A_2$, $M_{\mathrm{row}} = \max(A_1, A_2)$. Restrict the smooth f sums in K_0, K_S to $Nf > Y \geq 1$. The inherited positive input gives*

$$\begin{aligned} \frac{|K_0^{>Y}|}{D} &\ll_\epsilon (HD)^\epsilon B^2 \left\{ R + \frac{DR}{M_{\mathrm{row}} Y} + \frac{D^{2/3}R}{M_{\mathrm{row}}^{1/3} Y^{2/3}} \right\}, \\ K_S^{>Y} &= 0 \qquad (Y \geq 2D/U). \end{aligned} \tag{53}$$

If $R \ll H$, a sufficient target-size tail cutoff is

$$Y \geq \max\left\{ 1, \frac{2D}{U}, \frac{DR}{M_{\mathrm{row}} H}, \frac{DR^{3/2}}{M_{\mathrm{row}}^{1/2} H^{3/2}} \right\}.$$

This is a coarse inherited sufficient bound, not signed cancellation or an optimal localization. The smooth incidence divisor f is not the original column-pair gcd h.

Theorem 21 (A conditional input diagnostic)**.** *Fix S and $0 < \alpha < 1/2$. If for every $\epsilon > 0$,*

$$\mathfrak{d}_2(T) \ll_{\epsilon,K,S} T^{-\alpha+\epsilon}$$

holds for all sufficiently large real T, then

$$\zeta_K(s) \neq 0 \qquad (\mathrm{Re}\, s > \max\{1-\alpha, 1/2\},\ s \neq 1).$$

This is a strict-region implication from an explicitly hypothetical input, not a new zero-free theorem or a necessary condition for the actual signed target.

With the current physical input the pure witness has the sufficient PIT majorant 19/24, and the mixed witness retains the smaller 7/9 from the minimum. Independent scalar powers $U^{-1/12}$ and $U^{-1/18}$ would suffice for those majorants, with compatible endpoint control; their all-scale mean-square versions would entail the strict strips 11/12 and 17/18 above. None is asserted available. The old pure 5/6 budget and $U^{-1/6}$ input, with conditional strip 5/6, are historical values for the former operator bound. Proposition 23 gives the current global 9/8 comparison, and Proposition 24 the scoped $A_6 = 1$ comparison 13/12. The target, principal/short assembly beyond the stated scopes, independent determinant/zero-location bridges and full analytic/category Lean formalization remain open.

21 A sieve comparison indexed by the physical radicand

This section applies a new external preprint input to the original rows. It is an application of that theorem, not a new proof of a large sieve, a signed target estimate, or an analytic Lean result. Throughout, $K = \mathbb{Q}(\omega)$, S is fixed and contains the primes above $2, 3$ and the original conductor exclusions, and ν is the original fixed finite-order Hecke column character. Let $D, H \geq 1$, $U \geq 1$, $U^2 \leq D$, and let W be one common possibly complex smooth profile supported on $[1, 2]$, zero at its endpoints. Write $B_W = \|W\|_\infty$. The estimates below concern the original unweighted literal rows, or already justified bounded natural multiplicities. They do not insert arbitrary positive row weights into an unweighted theorem.

Pinned external input. De Faveri's arXiv:2610.04045v1, Theorem 1.1, bounds the squared row norm $\Theta_n(A,B)$ for the Section 2.3 family of order-dividing-n Hecke characters, with both indices nth-power-free ideals prime to fixed S:

$$\Theta_n(A,B) \ll_\epsilon (AB)^\epsilon (A+B+A^{1-1/n}B^{2/n}+A^{2/n}B^{1-1/n}). \tag{54}$$

Here $n \geq 3$, K contains μ_n, and the chosen fixed S-integer ring has class number one. The constants may depend on the fixed family construction. We use $n=6$, not a conjectural $A+B$ bound; the additional μ_{2n} condition of that paper's Theorem 1.4 is not a hypothesis of Theorem 1.1. Set

$$\Phi_6(A,B) = A+B+A^{5/6}B^{1/3}+A^{1/3}B^{5/6},$$
$$\Phi_{\mathrm{all}}(H,D) = H+DH^{1/6}+H^{5/6}D^{1/3}+H^{1/3}D^{5/6}.$$

Proposition 22 (Exact raw-family identification)**.** *There is an allowed fixed choice of the data in* (54) *for which, for every good ideal* $\mathfrak{a}$*, with unique generator* $\alpha_\mathfrak{a} \equiv 1 \pmod 3$*,*

$$\chi_\mathfrak{a}(\mathfrak{m}) = \left(\frac{\alpha_\mathfrak{a}}{\mathfrak{m}}\right)_6$$

including the original zero at each common prime. In particular, the row index is the physical sixth-power-free ideal, including valuations $2,3,4,5$*, rather than its radical conductor.*

The extended proof specifies the data and removes the possible unit ambiguity. No extra column-ray selection is made. For the unique outside-S decomposition of an actual literal row,

$$\begin{aligned} u &= u_0 \alpha_\mathfrak{a} \alpha_{\mathfrak{r}_6}^6, & \mathfrak{a} &= \mathfrak{r}_1\mathfrak{r}_2^2\mathfrak{r}_3^3\mathfrak{r}_4^4\mathfrak{r}_5^5, \\ \chi_\mathfrak{m}(u) &= \chi_\mathfrak{m}(u_0)\chi_\mathfrak{a}(\mathfrak{m})\mathbf{1}_{(\mathfrak{m},\mathfrak{r}_6)=1}, & Nu &= Nu_0\, N\mathfrak{a}\, (N\mathfrak{r}_6)^6. \end{aligned} \tag{55}$$

The first five factors are squarefree and pairwise coprime; $\mathfrak{r}_6$ is arbitrary and may overlap them. All original union, common-prime, literal, ray and capacity restrictions remain.

Proposition 23 (Physical-row comparison)**.** *For the original raw sums* I_u *of* (6)*, and for the original residue-subtracted* $E_u = I_u - P_u$ *with fixed* ν, S*,*

$$\sum_{0<Nu\leq H} |I_u|^2 + \sum_{0<Nu\leq H} |E_u|^2 \ll_{\nu,S,\epsilon} B_W^2 (HD)^\epsilon \Phi_{\mathrm{all}}(H,D).$$

For $1 \leq H \leq D$ *one has*

$$\Phi_{\mathrm{all}}(H,D) \leq 4DH^{1/6}. \tag{56}$$

Thus the critical positive envelope at $H = D^{3/4}$ *is* $D^{9/8+\epsilon}$*, improving the previous* $7/6$ *comparison by* $1/24$ *while still exceeding the* $3/4$ *target by* $3/8$*.*

The agreement with the varying-profile obstruction of Section 16 concerns only a supremum-norm-controlled class allowing W_D to vary. It is not a lower bound or a sharpness assertion for one fixed W, nor a GRH counterexample.

Proposition 24 (The actual $A_6 = 1$ signed slice)**.** *For an original unweighted two-pivot block, retaining all its actual filters and the complete annular columns, the raw form* S_1^{AA} *of* (25) *satisfies*

$$|S_1^{AA}| \ll_{\nu,S,\epsilon} B_W^2 (HD)^\epsilon \Phi_6(H,D).$$

The same bound holds after summing the fixed-S parts of such rows: the D term costs a fixed polylogarithmic count, absorbed in $(HD)^\epsilon$*, not a convergent exponent-zero Euler sum. At* $H = D^{3/4}$ *the available power is* $13/12$*, an improvement of* $1/12$ *from the old* $7/6$ *majorant, but still* $1/3$ *above the target.*

The proof inserts (54) into the exact original common-divisor identity (27); $A_U(dn)$ is not factored and is not put into the rough short-coefficient domain. Under the additional high-window and capacity hypotheses of Section 20, its exact ordered rough correction gives

$$\left|\frac{K_0 - K_{\mathrm{all}}}{D}\right| \le |S_1^{AA}| + 236800 B_W^2 R_w. \tag{57}$$

This estimates the actual combined form through its exact identity; it does not take separate main-term triangles in K_0, K_{all}, and does not prove signed cancellation at the target scale.

Proposition 25 (Updated short positive certificate)**.** *On an original unweighted block, put $R = A_1A_2$, $M = \max(A_1, A_2)$ and $H_b = \min\{H/Nu_0, 8A_1A_2^2\}$ for fixed literal/S factor u_0. Empty blocks contribute zero; a nonempty block has $H_b \ge 1$. For $1 \le L \le D$ the actual squared row-operator norm has the bound*

$$\lambda_{\mathrm{now}}(L) \ll (HD)^{\epsilon} \min \begin{Bmatrix} R + \dfrac{LR}{M} + \dfrac{L^{2/3}R}{M^{1/3}}, & L + R^2, \\ \Phi_6(H_b, L) \end{Bmatrix}.$$

For $1/2 \le L < 1$ use the direct unit-column norm R_w. This new norm may be substituted into the mean-retaining joint Abel criterion of Proposition 19, with its lower endpoint intact. At $H = D^{3/4}$, $U \asymp D^{1/2}$, its pure witness costs $19/24$; at the displayed mixed witness the minimum retains the old $7/9$ bound, which beats the generic $19/24$.

In this positive certificate sufficient independent scalar discrepancy powers are respectively

$$U^{-1/12+\epsilon} \quad \text{and} \quad U^{-1/18+\epsilon}. \tag{58}$$

If imposed as all-scale mean-square inputs, these would entail the conditional zero-free strips $\Re s > 11/12$ and $\Re s > 17/18$ of Theorem 21. Neither input nor strip is proved here; they are not necessary conditions for the actual signed target. The endpoint and coherent $K_0 - J$ remain. Smooth-tail localization (Proposition 20) does not remove $f = 1$ or supply this estimate.

22 Coherent dual bands, actual cutoffs and arithmetic localization

The physical-row comparison in Section 21 leaves a signed problem. We now retain its exact primitive roots and sharp row capacity, prove quantitative finite-prefix and divisor-band comparisons, and localize a coherent sector that can already be controlled. The remaining unequal-norm signed moment is stated explicitly. These results do not improve the whole signed power $13/12$ or the global unsigned power $9/8$. The deep sieve input remains the pinned external theorem (54); its cubic specialization is used below.

22.1 The original form and the scope of the dual representation

Let $K = \mathbb{Q}(\omega)$, $\mathcal{O} = \mathbb{Z}[\omega]$ and $\delta = 1 + 2\omega = i\sqrt{3}$. The fixed set S contains the primes above $2, 3$ and the original exclusions. All columns are the complete good squarefree *ideal* annulus $D \le N\mathfrak{m} \le 2D$, $D > 1$. Distinct ideals of the same norm remain distinct. Put

$$\begin{aligned} A_U(\mathfrak{m}) &= \sum_{\substack{\mathfrak{r}\mathfrak{s}\mid\mathfrak{m} \\ (\mathfrak{r},\mathfrak{s})=1 \\ N\mathfrak{r}, N\mathfrak{s}\le U}} \mu(\mathfrak{r})\mu(\mathfrak{s}), \\ c_{\mathfrak{m}} &= A_U(\mathfrak{m})\nu(\mathfrak{m})R_{\mathfrak{m}}(u_0)W(N\mathfrak{m}/D). \end{aligned} \tag{59}$$

Here $1 \le U \le \sqrt{D}$, ν is the original fixed unitary finite-order column character, and W is one common possibly complex smooth profile supported on $[1,2]$, zero at its endpoints. Write $B_W = \|W\|_\infty$. For a good ideal $\mathfrak{v}$, let $\alpha_\mathfrak{v} \equiv 1 \pmod 3$ be its unique primary generator and set $R_\mathfrak{m}(\mathfrak{v}) = (\alpha_\mathfrak{v}/\mathfrak{m})_6$. The literal symbol is zero at a common prime. A primitive extension at an S-argument below is a different object, not restoration of a deleted raw row value.

Let $A_1, A_2 \ge 1$. Only the original unweighted two-pivot rows are used:

$$u = u_0 \alpha_{\mathfrak{v}_1} \alpha_{\mathfrak{v}_2}^2, \qquad A_i \le N\mathfrak{v}_i < 2A_i, \qquad N u_0 N\mathfrak{v}_1 (N\mathfrak{v}_2)^2 \le H.$$

The ideals $\mathfrak{v}_1, \mathfrak{v}_2$ are good, squarefree and coprime; all actual literal and ray restrictions remain. The other four factors are the unit ideal. The fixed u_0 is literal unit/S-supported, without good-prime support. Extra good frozen factors, a sum over varying u_0, and arbitrary row weights are not included. Write $R = A_1 A_2$ and let R_w be the actual number of rows. The normalization is

$$\begin{aligned} \mathcal{P}_D &= D S_1^{AA} \\ &= \sum_{(\mathfrak{v}_1,\mathfrak{v}_2)} \sum_{\text{actual } (\mathfrak{m},\mathfrak{n})=1} c_\mathfrak{m} \overline{c_\mathfrak{n}} R_\mathfrak{m}(\mathfrak{v}_1 \mathfrak{v}_2^2) \overline{R_\mathfrak{n}(\mathfrak{v}_1 \mathfrak{v}_2^2)}. \end{aligned} \tag{60}$$

The original Möbius signs are already in the two A_U coefficients; no additional $\mu(\mathfrak{mn})$ is inserted. This raw coprime form is not a positive energy or principal-term assembly.

22.2 Primitive normalization and an exact finite-cutoff kernel

For each source class A, use the fixed decomposition $\widetilde{\chi}_\mathfrak{m} = \xi_\mathfrak{m} \psi_A$, $\mathrm{cond}(\xi_\mathfrak{m}) = \mathfrak{m}$, and $\beta_\mathfrak{m} = \epsilon(\xi_\mathfrak{m}) \psi_A(\mathfrak{m})$. Individual ξ and ψ may have nonzero fixed angular and radial unitary types; they are not declared sixth-order. The finite source reciprocity factors and the *whole* original row-ray projector are expanded in their complete finite character basis, with coefficients $h_{AB\rho}(\mathfrak{v}_2)$. On coprime class-$A, B$ columns the resulting primitive character is

$$\begin{aligned} &\sigma = \mathrm{prim}(\rho \psi_A \overline{\psi_B}), \quad \mathfrak{f}_S = \mathrm{cond}(\sigma), \\ &\eta = \xi_\mathfrak{m} \overline{\xi_\mathfrak{n}} \sigma, \qquad\qquad F = N\mathfrak{f}_S \, N\mathfrak{m} \, N\mathfrak{n}. \end{aligned} \tag{61}$$

Its total infinity type is zero. The actual S conductor, including any drop, is retained.

Proposition 26 (Primitive roots with their types retained)**.** *For a primitive unitary Hecke character $\chi((\alpha)) = \theta(\alpha)\,\mathrm{phase}(\alpha)^k N(\alpha)^{it}$, with conductor (γ), let*

$$\tau_\gamma(\theta) = \sum_{x \bmod \gamma} \theta(x) \exp\bigl(2\pi i \,\mathrm{Tr}(x/(\delta\gamma))\bigr).$$

Then

$$\epsilon(\chi) = (3N\mathfrak{f})^{it} i^{-|k|} \,\mathrm{phase}(\delta\gamma)^k \frac{\tau_\gamma(\theta)}{\sqrt{N\mathfrak{f}}}. \tag{62}$$

The unit-conductor/principal formula has its usual meromorphic terms. For pairwise coprime primitive conductors,

$$\epsilon\Bigl(\prod_j \chi_j\Bigr) = i^{\sum_j |k_j| - |\sum_j k_j|} \prod_j \epsilon(\chi_j) \prod_{i \ne j} \chi_i(\mathfrak{f}_j). \tag{63}$$

Consequently the actual character in (61) *satisfies*

$$\begin{aligned} \epsilon(\eta) &= \kappa_{AB\rho} u_\mathfrak{m} \overline{v_\mathfrak{n}}, \\ u_\mathfrak{m} &= \beta_\mathfrak{m} \xi_\mathfrak{m}(\mathfrak{f}_S) \rho(\mathfrak{m}), \\ v_\mathfrak{n} &= \beta_\mathfrak{n} \xi_\mathfrak{n}(\mathfrak{f}_S) \overline{\rho(\mathfrak{n})}. \end{aligned} \tag{64}$$

The scalar $\kappa_{AB\rho}$ is fixed. These are two distinct unitary column gauges. The primitive values $\sigma(\mathfrak{e})$ and $\sigma(\mathfrak{h})$ at S, including zero, remain separately in the kernel; they cannot be replaced by ρ there.

This closes the normalization hypothesis for these stated conventions. A bounded, invertible change of phases gives no power saving by itself.

For a fixed actual $\mathfrak{v}_2$, define the exact integer endpoints

$$a = \lceil A_1 \rceil, \qquad b = \min\{\lceil 2A_1 \rceil - 1, \lfloor H/(Nu_0(N\mathfrak{v}_2)^2) \rfloor\}.$$

An empty interval contributes zero. Choose one fixed nondecreasing smooth step s_0, equal to zero on $(-\infty, 0]$ and one on $[1, \infty)$, and put

$$G_{\mathfrak{v}_2}(x) = s_0(4(A_1 x - a + 1/2))s_0(4(b + 1/2 - A_1 x)). \tag{65}$$

It is the exact row indicator at every ideal norm node, has amplitude at most one, support in $[1/2, 5/2]$ and total variation at most two. Its derivatives cost powers of A_1; it is not the fixed column W. Set $C_0 = 3/(4\pi^2)$ and

$$\dot{G}(X) = C_0^{-1/2} \int_0^\infty G(t) J_0(2\sqrt{Xt/C_0})\, dt. \tag{66}$$

Proposition 27 (The actual signed kernel before absolute norms)**.** *For every nonzero class/ray component, the good conductor of η is $\mathfrak{mn}$, so η is nonprincipal. Put $P_S = \prod_{p \in S} p$ and*

$$L_{\mathfrak{e}} = \frac{A_1}{(N\mathfrak{k})^2 N\mathfrak{l} N\mathfrak{e}}, \qquad K_{\mathfrak{l}} = \sqrt{b/N\mathfrak{l}}.$$

Squarefree-row inversion, finite S deletion and primitive Poisson give (60) *by summing the following kernel:*

$$\begin{aligned} &c_{\mathfrak{m}}\overline{c_{\mathfrak{n}}} R_{\mathfrak{m}}(\mathfrak{v}_2^2)\overline{R_{\mathfrak{n}}(\mathfrak{v}_2^2)} h_{AB\rho}(\mathfrak{v}_2) \\ &\quad \cdot \mu(\mathfrak{k})\mu(\mathfrak{l})\mu(\mathfrak{e})\eta(\mathfrak{k}^2\mathfrak{l}\mathfrak{e})\epsilon(\eta)\frac{L_{\mathfrak{e}}}{\sqrt{F}} \sum_{\mathfrak{y}} \dot{G}_{\mathfrak{v}_2}(L_{\mathfrak{e}} N\mathfrak{y}/F)\overline{\eta(\mathfrak{y})}. \end{aligned} \tag{67}$$

The outer sums are over every actual $\mathfrak{v}_2$, every class/ray component and original coprime column pair, all $\mathfrak{l} \mid \mathfrak{v}_2$ with $N\mathfrak{l} \le b$, all good squarefree $\mathfrak{k}$ with $(\mathfrak{k}, \mathfrak{v}_2) = 1$ and $N\mathfrak{k} \le K_{\mathfrak{l}}$, and every $\mathfrak{e} \mid P_S$. The dual sum is over all *nonzero integral ideals, including S-supported ones. It converges at each fixed parameter set with the actual derivative constants of G. The original pre-S cutoff is never extended in this identity.*

All signs, means, product norms and raw zeros occur before a triangle. A higher induced modulus does not have a unit Gauss root automatically: its nonprimitive additive frequencies may vanish.

22.3 What finite prefixes and actual divisor bands now satisfy

Write

$$\begin{aligned} \Phi(Z, L) &= Z + LZ^{1/6} + Z^{5/6}L^{1/3} + Z^{1/3}L^{5/6}, \\ I(t) &= \sum_{j \ge 0} \frac{t^j}{(j!)^2}. \end{aligned} \tag{68}$$

Let $\mathcal{P}_{\le X}$ retain all terms of (67) with $N\mathfrak{y} \le X$, including every source/ray/S component.

Proposition 28 (All-component exact finite prefixes)**.** *For every finite $X \geq 1$,*

$$\frac{|\mathcal{P}_{\leq X}|}{D} \ll B_W^2 (DXA_2)^{\epsilon} \frac{A_1 A_2}{D} \Phi(X, D). \tag{69}$$

A separate exact all-orders estimate is

$$|\mathcal{P}_{\leq X}| \ll B_W^2 (DXA_2)^{\epsilon} A_1 A_2 \Phi(X, D) I\left(\frac{(5/2)A_1 X}{C_0 D^2}\right). \tag{70}$$

No completed Euler model, low-frequency approximation or $X \leq \sqrt{H}$ hypothesis is used in (69)*. On a fixed same-source-class/$\rho = 1$ component, the additional critical-half band estimate is*

$$\frac{|\mathcal{P}_{C,1}(Y, 2Y]|}{D} \ll B_W^2 (DYA_2)^{\epsilon} A_2 \sqrt{A_1/Y}\, \Phi(Y, D)(1 + \log(2A_1))^2. \tag{71}$$

Constants depend on the fixed S, ray/source data, ν, step and ϵ, not on any hidden uniform smoothness of the sharp G.

At $H \asymp D^{3/4}$, the mixed witness $(A_1, A_2) \asymp (D^{5/12}, D^{1/6})$ has a target-sized full finite prefix $X \asymp D$. The pure prefix at $X = D$ has upper power $11/12$. At $X = D^{3/8}$ the pure/mixed powers are $13/16$ and $31/48$. These growing bounds cannot be sent to $X = \infty$.

Proposition 29 (Finite transfer of the actual cutoff and profile)**.** *Let $X \geq 2$, $1 \leq U \leq X$, and let $K = (K_{\mathfrak{d}\mathfrak{a}})$ be a finite matrix indexed by good squarefree ideals of norms at most X, zero unless $(\mathfrak{d}, \mathfrak{a}) = 1$ and $N\mathfrak{d} N\mathfrak{a} \leq X$. All additional masks may remain in K. Set*

$$\begin{aligned} G_{\mathcal{I}\mathcal{J}} &= \max_{\mathfrak{d} \in \mathcal{I}} 3^{\omega(\mathfrak{d})} \max_{\mathfrak{a} \in \mathcal{J}} 3^{\omega(\mathfrak{a})}, \\ c_P &= 1/\log(4X), \qquad L_T = \operatorname{asinh}(T/c_P)/\pi. \end{aligned}$$

Empty matrices are zero. For every finite $T \geq 8X$,

$$\|[A_U(\mathfrak{d}\mathfrak{a}) K_{\mathfrak{d}\mathfrak{a}}]\|_{\mathrm{op}} \leq G_{\mathcal{I}\mathcal{J}} \left(e^2 L_T^2 + \frac{32 X^{3/2}}{T}\right) \|K\|_{\mathrm{op}}. \tag{72}$$

For $T = (4X)^4$ this is $O_\epsilon(X^\epsilon \log^2(4X))\|K\|$. The common complex multiplier $W(N\mathfrak{d}N\mathfrak{a}/D)$ costs

$$\begin{aligned} M(W) &= \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{W}(it)|\, dt, \qquad \widehat{W}(s) = \int_0^\infty W(x) x^s \frac{dx}{x}, \\ M(W) &\leq \frac{\log 2}{\pi} (B_W + \|xW'(x) + x^2 W''(x)\|_\infty). \end{aligned} \tag{73}$$

Two occurrences in a bilinear/Gram estimate cost the square of the transfer norm. This is not $A_U(\mathfrak{d}\mathfrak{a}) = A_U(\mathfrak{d}) A_U(\mathfrak{a})$ or coefficient shrinkage.

Proposition 30 (A cubic comparison for an actual auxiliary band)**.** *Fix one same-source-class/$\rho = 1$ component of the auxiliary individual-column extension of* (67)*. Insert the exact coprime identity $\sum_{\mathfrak{d} | (\mathfrak{m}, \mathfrak{n})} \mu(\mathfrak{d})$, and retain the* identity-source-class *divisors $E < N\mathfrak{d} \leq 2E$, $1 \leq E \leq D/2$. Denote the resulting exact signed dual band by $\mathcal{P}^{\mathrm{id}}_{E,C}(Y, 2Y]$. Put $L = D/E$ and $\Theta_3(E, L) = E + L + (EL)^{2/3}$. For fixed $0 < \lambda < 1/2$,*

$$\begin{aligned} \frac{|\mathcal{P}^{\mathrm{id}}_{E,C}(Y, 2Y]|}{D} \ll\ & M(W)^2 (DYA_2)^{\epsilon} A_2 A_1^{1-\lambda} D^{2\lambda - 1} Y^{-\lambda} \\ & \cdot \min\{\Phi(Y, L),\ (Y/E)\Theta_3(E, L)\}. \end{aligned} \tag{74}$$

The arithmetic identity behind the new branch is

$$\beta_{\mathfrak{d}\mathfrak{a}} = \epsilon(\widetilde{\chi}_{\mathfrak{d}})\beta_{\mathfrak{a}}\alpha(1,C)\widetilde{\chi}_{\mathfrak{a}}(\mathfrak{d})^2, \qquad [\mathfrak{d}] = 1, \quad [\mathfrak{a}] = C. \tag{75}$$

It exposes the physical cubic residue matrix in $\mathfrak{d}, \mathfrak{a}$. The external input is de Faveri, Theorem 1.1 and Example 2.2, now at $n = 3$, not a conductor-parametric sieve.

This improves a *scoped norm comparator*: at $E \asymp D^{2/3}$ the second branch is $O(Y)$. It still uses $|\mu(\mathfrak{d})| \le 1$ at its final Cauchy step. Such $\mathfrak{d}$ are auxiliary divisors, not the original column-pair gcd h. All divisor classes, small $\mathfrak{d}$ and the full high-dual sum are not bounded here.

22.4 Coherent means that can be localized, and those that remain

Proposition 31 (The full equal-norm sector)**.** *Original coprime squarefree columns with $N\mathfrak{m} = N\mathfrak{n} = N > 1$ satisfy $\mathfrak{n} = \overline{\mathfrak{m}}$, with rational-squarefree N supported on split rational primes. The original good masks require* both *primes above each such rational prime to avoid S; S need not be enlarged or conjugation-stable. There are $O(D)$ ordered annular pairs. Define S_{eq} as the normalized contribution to S_1^{AA}, namely D^{-1} times their full primal contribution to $\mathcal{P}_D$ in* (60)*. Then*

$$|S_{\mathrm{eq}}| \ll B_W^2 R_w D^{\epsilon} \ll B_W^2 R (DH)^{\epsilon}. \tag{76}$$

Before the factor D^{-1}, their ordered orientation sum at norm N is

$$\begin{gathered} a_U(N)^2 |W(N/D)|^2 \prod_{p|N} \{a_p \chi_p(t) + \overline{a_p \chi_p(t)}\}, \\ t = N u_0 N \mathfrak{v}_1 (N\mathfrak{v}_2)^2, \end{gathered} \tag{77}$$

where $a_p = \nu(\pi_p)\overline{\nu(\bar{\pi}_p)}$, $\chi_p(t) = (t/\pi_p)_6$, and $a_U(N) = \sum_{rs|N,(r,s)=1,r,s\le U} \mu_{\mathbb{Z}}(r)\mu_{\mathbb{Z}}(s)$. This includes all ordered orientations and every raw collision zero. For the all-component primitive-ideal prefix/band,

$$\frac{|\mathcal{P}_{\mathrm{eq}}(\le Y)| + |\mathcal{P}_{\mathrm{eq}}(Y, 2Y]|}{D} \ll B_W^2 (DY A_2)^{\epsilon} RY/D. \tag{78}$$

The signed complement is bounded by the whole primal term minus that prefix, not by an absolute high-mode summation.

Since $R \ll H$ on nonempty blocks, the desired full fixed-W target is equivalent, up to (76), to that for unequal norms. Its desired constants may depend on fixed W or appropriate specified seminorms. Formula (77) is neither identically zero nor positive in general.

Proposition 32 (Literal-ray recombination in the pure sector)**.** *In the pure case $\mathfrak{v}_2 = \mathcal{O}$, choose an S-supported ray modulus $\mathfrak{c} = (\gamma_c)$ divisible by* (6)*. For an original coprime pair put $\mathfrak{q} = \mathfrak{m}\mathfrak{n}$, $Q = N\mathfrak{q}$, $\gamma_q = \alpha_{\mathfrak{m}}\alpha_{\mathfrak{n}}$, and let $\vartheta_{\mathfrak{q}}(x) = (x/\mathfrak{m})_6\overline{(x/\mathfrak{n})_6}$ be the primitive finite character modulo $\mathfrak{q}$. Let $P_{\mathfrak{k}^2}$ be the full literal primary/good-S/ray indicator on $\mathcal{O}/\mathfrak{c}$ after $v_1 = k^2 z$, with finite additive transform $\widehat{P}$. The exact common-modulus Fourier coefficient is*

$$\widehat{F}_{\mathfrak{k}^2}(\beta) = \tau_{\mathfrak{q}}\vartheta_{\mathfrak{q}}(\gamma_c)\overline{\vartheta_{\mathfrak{q}}(\beta)}\widehat{P}_{\mathfrak{k}^2}(\beta\gamma_q^{-1} \bmod \mathfrak{c}). \tag{79}$$

Its literal-lattice radial Poisson prefactor is $L_{\mathfrak{k}}/(N\mathfrak{c}Q)$, $L_{\mathfrak{k}} = A_1/(N\mathfrak{k})^2$. The S-local Fourier-mean sublattice $\beta = \gamma_c\zeta$ has coefficient $r_{\mathfrak{k}^2}\tau_{\mathfrak{q}}\overline{\vartheta_{\mathfrak{q}}(\zeta)}$, where $r = \sum P$. The six-unit projection annihilates every nontrivial unit restriction and retains the unit-trivial branch, whose condition is

$$N\mathfrak{m} \equiv N\mathfrak{n} \pmod{36}. \tag{80}$$

For the basic S above $2,3$ with no extra ray condition, $\mathfrak{c} = (6)$ has $r = 3$ and ideal coefficient $6r/N\mathfrak{c} = 1/2$. This is not the global $\beta = 0$ mode, which vanishes. All S-supported dual ideals remain, with values defined by any generator once the character is unit-trivial.

Literal additive bands and the earlier primitive-ideal bands have different fixed-S frequency support/rescalings; no unmatched windows are identified. Unequal norms congruent modulo 36 and all other literal residue channels remain. Finite ray recombination supplies no automatic zero mean or signed target estimate.

Remaining arithmetic obligation. A finite-prefix comparison, the sparse equal-norm sector and the cubic auxiliary-band comparison do not bound the complete signed unequal-norm kernel. The required additional estimate must retain all small divisors/source classes and the literal S mean together with complementary residues before the final conductor/divisor/dual triangle. No desired positivity, energy lower bound, new zero-free strip, determinant comparison, analytic Lean or GRH claim follows.

23 An arithmetic Gauss tail with the original finite cutoff

The coherent kernel of Section 22 contains a difficult root-weighted mean. A large part of its *short-divisor expansion* can now be bounded by retaining a cubic interaction before taking a triangle. This gives a signed cutoff comparison, not a smaller norm for the original coefficients. A direct exact-discrete-frequency version is included as well. The remaining small-short-divisor moment and the separate discrete-frequency error are stated below. The whole signed power 13/12 and global unsigned power 9/8 do not change.

23.1 Original objects and a fixed common profile

Keep $K = \mathbb{Q}(\omega)$, $\mathcal{O} = \mathbb{Z}[\omega]$, $\delta = 1 + 2\omega$, and the fixed exclusions S of Section 22.1. A good ideal is prime to S. All columns are the complete good squarefree ideal annulus $D \le N\mathfrak{m} \le 2D$, $D > 1$. Equal norms do not identify ideals. For a real cutoff $1 \le T \le \sqrt{D}$ put

$$\begin{aligned} A_T(\mathfrak{m}) &= \sum_{\substack{\mathfrak{a}\mathfrak{b}\mid\mathfrak{m}\\(\mathfrak{a},\mathfrak{b})=1\\ N\mathfrak{a},N\mathfrak{b}\le T}} \mu(\mathfrak{a})\mu(\mathfrak{b}),\\ c_f(\mathfrak{m}) &= f(\mathfrak{m})\nu(\mathfrak{m})R_{\mathfrak{m}}(u_0)W(N\mathfrak{m}/D). \end{aligned} \tag{81}$$

Here ν is the original fixed unitary finite-order character; u_0 is the fixed literal unit/S-supported factor with no good support. The actual coefficient is c_{A_U}, $1 \le U \le \sqrt{D}$. The function W is one fixed, possibly complex, smooth common profile supported on $[1,2]$, zero at the endpoints. Define

$$\begin{aligned} B_W &= \|W\|_\infty, \qquad \widehat{W}(t) = \int_0^\infty W(x)x^{-it}\,\frac{dx}{x},\\ \mathcal{M}(W) &= \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{W}(t)|\,dt. \end{aligned} \tag{82}$$

This explicit seminorm is finite and controls B_W. It is a seminorm of the column profile, not of the sharp nodal row function.

Only the original unweighted *pure* two-pivot block is used: $A_2 = 1$, $\mathfrak{v}_2 = \mathcal{O}$, and the other four factors are units. The first row is good and squarefree, with the original literal/ray conditions and capacity. Its integer endpoints are

$$a = \lceil A_1 \rceil, \qquad b = \min\{\lceil 2A_1 \rceil - 1, \lfloor H/Nu_0 \rfloor\}, \qquad K = \sqrt{b}. \tag{83}$$

An empty interval contributes zero throughout. Otherwise $b \geq a \geq 1$. Use the exact G of (65), its transform $\dot{G}$ in (66), and $C_0 = 3/(4\pi^2)$. In particular $0 \leq G \leq 1$, $\operatorname{supp} G \subset [1/2, 5/2]$, and $\int G \leq 2$. The original *pre-S* cutoff K is never recomputed after another deletion.

Let $\mathcal{I}$ be the original ordered coprime column pairs. One may also impose the *original* unequal-norm projection; the same choice is then used in every expression below. For each source pair A, B and original ray character ρ use the primitive η, $\mathfrak{f}_S$ and F of (61). Thus $F = N\mathfrak{f}_S N\mathfrak{m} N\mathfrak{n}$, and η has total infinity type zero. Put

$$\begin{aligned} T_q &= S \cup \operatorname{supp}(\mathfrak{m}\mathfrak{n}), \\ Q_s^{T_q}(K;\eta^2) &= \sum_{\substack{\mathfrak{k} \text{ good,SF} \\ N\mathfrak{k} \leq K,\ (\mathfrak{k},T_q)=1}} \frac{\mu(\mathfrak{k})\eta(\mathfrak{k})^2}{(N\mathfrak{k})^s}, \\ P_S(\eta) &= \sum_{\mathfrak{e} \mid P_S} \frac{\mu(\mathfrak{e})\eta(\mathfrak{e})}{N\mathfrak{e}}, \qquad P_S = \prod_{p \in S} p. \end{aligned} \tag{84}$$

The symbol $(\mathfrak{k}, T_q) = 1$ means avoidance of every prime in that set. Squaring η never restores a deleted prime. Values at $\mathfrak{e}$ and at any later S-supported frequency are separate primitive values, including zero.

23.2 The actual retained mean and its finite leading model

In the exact primitive kernel (67), write each dual ideal uniquely as $\mathfrak{y} = \mathfrak{h}\mathfrak{r}\mathfrak{s}^6$: $\mathfrak{h}$ is S-supported, $\mathfrak{r}$ is good and sixth-free, and $\mathfrak{s}$ is good. Overlap of $\mathfrak{r}, \mathfrak{s}$ is allowed. For $X \geq 64$ set

$$V = X^{1/6}/2, \qquad b_{\mathfrak{w}} = (X/N\mathfrak{w})^{1/6}, \qquad t_{\mathfrak{w}} = \left((N\mathfrak{w})^{-1/6} - 1/2\right)_+. \tag{85}$$

The top band is $V < N\mathfrak{s} \leq 2V$, $N\mathfrak{y} \leq X$. Replacing its $\mathfrak{s}$ count by its principal fixed-ray density defines the retained mean M_{top}. This operation keeps, rather than deletes, its coherent main term. The selected branches are $\rho^6 = 1$. At this top window $N\mathfrak{h}N\mathfrak{r} < 64 < 7^6$, so $\mathfrak{w} = \mathfrak{h}\mathfrak{r}$ runs over *all* integral ideals of norm below 64, including S-supported ideals.

Set

$$\kappa_S = \frac{\pi}{3\sqrt{3}} \prod_{p \in S} (1 - (Np)^{-1}), \qquad g(\mathfrak{m}) = \prod_{p \mid \mathfrak{m}} (1 - (Np)^{-1}). \tag{86}$$

On original coprime support the density is $\kappa_S g(\mathfrak{m}) g(\mathfrak{n})$. For $\theta = (A, B, \rho, \mathfrak{m}, \mathfrak{n}, \mathfrak{e}, \mathfrak{w})$ in the preceding sets, with $\rho^6 = 1$, define

$$\begin{aligned} d_\theta[f_1, f_2] = \frac{\kappa_S A_1}{D} h_{AB\rho}(\mathcal{O}) c_{f_1}(\mathfrak{m}) \overline{c_{f_2}(\mathfrak{n})} \frac{g(\mathfrak{m})g(\mathfrak{n})\epsilon(\eta)}{\sqrt{F}} \\ \cdot \frac{\mu(\mathfrak{e})\eta(\mathfrak{e})}{N\mathfrak{e}} \overline{\eta(\mathfrak{w})}, \qquad \lambda_\theta = \frac{A_1 N\mathfrak{w}}{N\mathfrak{e}F}. \end{aligned} \tag{87}$$

The exact definition of the normalized retained mean is

$$\mathcal{M}_{\mathrm{top}} := \frac{M_{\mathrm{top}}}{D} = \sum_{\theta} d_\theta[A_U, A_U] \sum_{\substack{\mathfrak{k}\ \text{good,SF} \\ N\mathfrak{k}\le K,\ (\mathfrak{k},T_q)=1}} \frac{\mu(\mathfrak{k})\eta(\mathfrak{k})^2}{(N\mathfrak{k})^2} \cdot \int_V^{b_{\mathfrak{w}}} \dot{G}\left(\frac{\lambda_\theta t^6}{(N\mathfrak{k})^2}\right) dt. \tag{88}$$

All sums here are finite. In particular there is no $\mathfrak{k} \mid \mathfrak{w}$ or $(\mathfrak{k},\mathfrak{w}) = 1$ condition. The finite original source/ray coefficients $h_{AB\rho}(\mathcal{O})$ have not been centered. An argument at $\mathfrak{e}$ is not divided by a value at $\mathfrak{w}$.

Put $g_0 = C_0^{-1/2} \int G$ and define

$$\mathcal{L}_0[f_1, f_2] = g_0 X^{1/6} \sum_{\theta} d_\theta[f_1, f_2]\, t_{\mathfrak{w}} Q_2^{T_q}(K; \eta^2). \tag{89}$$

Replacing both column functions here is an explicitly defined auxiliary model, not a change to the original primal data.

Proposition 33 (Matched leading model with its original cutoff)**.** *For the actual coefficients and every finite $X \ge 64$,*

$$|\mathcal{M}_{\mathrm{top}} - \mathcal{L}_0[A_U, A_U]| \ll B_W^2 (DX)^{\epsilon} \frac{A_1^2 X^{7/6}}{D^2}. \tag{90}$$

The constant uses no derivative bound for the sharp G. Both interval endpoints and the original finite K occur in $\mathcal{L}_0$. This estimates the model error, not the leading signed model.

23.3 The root is an incomplete cubic Gauss weight

For each good ideal use its primary generator $\alpha_{\mathfrak{a}} \equiv 1 \pmod 3$ and write $\chi_{j,\mathfrak{a}}$ for its positive residue character of order $j = 3, 6$. For good squarefree $\mathfrak{m}$ put

$$\begin{aligned} G_{\mathfrak{m}} &= \frac{1}{\sqrt{N\mathfrak{m}}} \sum_{x \bmod \alpha_{\mathfrak{m}}} \chi_{3,\mathfrak{m}}(x) \exp\!\left(2\pi i \operatorname{Tr} \frac{x}{\delta \alpha_{\mathfrak{m}}}\right), \\ g_{\mathfrak{m}}(z) &= \overline{\chi_{3,\mathfrak{m}}(z)} G_{\mathfrak{m}}. \end{aligned} \tag{91}$$

The second expression is the normalized finite Gauss sum at frequency z, including zero when $(\mathfrak{m}, z) > 1$. For $\mathfrak{q} = \mathfrak{m}\mathfrak{n}$ put $\theta_q = \chi_{6,\mathfrak{m}}\overline{\chi_{6,\mathfrak{n}}}$. Let θ_S be the *actual finite S part* of the total type-zero η, at its own primitive conductor $\mathfrak{f}_S = (\gamma_S)$. Then

$$\mathcal{S}_{\mathfrak{m},\mathfrak{n},\rho} = \theta_S(\alpha_{\mathfrak{m}}\alpha_{\mathfrak{n}}) \theta_q(\gamma_S) \frac{\tau_{\gamma_S}(\theta_S)}{\sqrt{N\mathfrak{f}_S}}. \tag{92}$$

The unit-conductor convention is one. This scalar has modulus one; it is not obtained by discarding the infinity type of a constituent. The accepted Jacobi character in (28) gives $\vartheta = \Psi/\sqrt{N}$, with conductor (2δ) of norm 12 and nonzero infinity type. The exact root formula is

$$\begin{aligned} \epsilon(\eta) &= \mathcal{S}_{\mathfrak{m},\mathfrak{n},\rho} \Gamma(\alpha_{\mathfrak{m}}\alpha_{\mathfrak{n}}) \mu(\mathfrak{m}) \mu(\mathfrak{n}) \vartheta(\mathfrak{m}) \overline{\vartheta(\mathfrak{n})}\, \overline{G_{\mathfrak{m}}} G_{\mathfrak{n}}, \\ \Gamma(a + b\omega) &= \tfrac{1}{2}\{1 + i^{-b} + i^{a} + i^{b-a}\}. \end{aligned} \tag{93}$$

No extra quadratic sign is inserted. In the original kernel, $\eta(\mathfrak{k}^2)$ and the second-row phase, when present, produce the raw cubic frequency $\overline{g_{\mathfrak{m}}(\alpha_{\mathfrak{k}}\alpha_{\mathfrak{v}_2})} g_{\mathfrak{n}}(\alpha_{\mathfrak{k}}\alpha_{\mathfrak{v}_2})$; here $\mathfrak{v}_2 = \mathcal{O}$. This incomplete weighted frequency set is not a complete residue average.

23.4 A signed short-divisor tail from a physical cubic matrix

For squarefree $\mathfrak{m}$ there is an exact sign-preserving assignment identity:

$$\mu(\mathfrak{m})A_T(\mathfrak{m}) = \sum_{\substack{\mathfrak{abc}=\mathfrak{m}\\ N\mathfrak{a},N\mathfrak{b}\le T}} \mu(\mathfrak{c}). \tag{94}$$

The factors are pairwise coprime and squarefree. For $Y \le U$ the difference of cutoff boxes has two disjoint pieces. After interchanging $\mathfrak{a},\mathfrak{b}$ in the second and setting $\mathfrak{z} = \mathfrak{bc}$, the exact coefficient is

$$T_U(\mathfrak{z}) + T_Y(\mathfrak{z}), \qquad T_L(\mathfrak{z}) = \sum_{\substack{\mathfrak{b}|\mathfrak{z}\\ N\mathfrak{b}\le L}} \mu(\mathfrak{z}/\mathfrak{b}). \tag{95}$$

It is independent of $\mathfrak{a}$. The conjugate Gauss CRT identity retains the cross interaction

$$\overline{G_{\mathfrak{az}}} = \overline{G_{\mathfrak{a}}G_{\mathfrak{z}}}\chi_{3,\mathfrak{a}}(\alpha_{\mathfrak{z}}), \qquad (\mathfrak{a},\mathfrak{z}) = 1. \tag{96}$$

The raw cubic entry is zero at a common prime.

The deep input is explicitly external: de Faveri, arXiv:2610.04045v1, Theorem 1.1 and Example 2.2, at $n = 3$. For the physical primary cubic matrix on good squarefree ideals of norms comparable to A, Z it gives

$$\|[\chi_{3,\mathfrak{a}}(\alpha_{\mathfrak{z}})]\|_{\mathrm{op}}^2 \ll_\epsilon (AZ)^\epsilon\{A + Z + (AZ)^{2/3}\}. \tag{97}$$

Both indices are actual ideals and hence cube-free, not arbitrary primitive-conductor labels. This is the same pinned input as the cubic comparison in Proposition 30.

Theorem 34 (Actual Gauss-weighted cutoff tail)**.** *For any bounded multiplicative ideal function λ on good ideals, with $|\lambda| \le 1$, and $1 \le Y \le U \le \sqrt{D}$,*

$$\begin{aligned}\left|\sum_{\mathfrak{m}\ \mathrm{good,SF}} \frac{\mu(\mathfrak{m})(A_U - A_Y)(\mathfrak{m})\overline{G_{\mathfrak{m}}}\lambda(\mathfrak{m})}{\sqrt{N\mathfrak{m}}} W(N\mathfrak{m}/D)\right| \\ \ll_\epsilon \mathcal{M}(W)D^\epsilon\{D^{1/3} + \sqrt{D/Y}\}.\end{aligned} \tag{98}$$

The constant is uniform in the conductor or raw deletion set in λ. For the actual pure leading model, with every original finite K, class/ray/root/S component and coprime pair retained,

$$|\mathcal{L}_0[A_U, A_U] - \mathcal{L}_0[A_Y, A_Y]| \ll B_W\mathcal{M}(W)D^\epsilon A_1 X^{1/6}\{Y^{-1/2} + D^{-1/6}\}. \tag{99}$$

The original unequal-norm projection is also allowed.

This uses the cubic interaction before the short-divisor triangle. It does not assert pointwise or coefficient-norm shrinkage of A_U, nor a separate cancellation theorem for $\mu(\mathfrak{c})$.

23.5 What has been reduced, and what remains

Combining Proposition 33 and Theorem 34 gives

$$\begin{aligned}|\mathcal{M}_{\mathrm{top}} - \mathcal{L}_0[A_Y, A_Y]| \ll (DX)^\epsilon \Bigg\{ B_W^2 \frac{A_1^2 X^{7/6}}{D^2} \\ + B_W\mathcal{M}(W)A_1 X^{1/6}(Y^{-1/2} + D^{-1/6})\Bigg\}.\end{aligned} \tag{100}$$

At $A_1 \asymp H = D^{3/4}$, $X = D^{3/8}$ and $Y = \min\{U, D^{1/8}\}$, the errors are target-sized. If $Y = U$ the coefficient tail is exactly zero; otherwise the three displayed powers are $-1/16$, $3/4$ and $31/48$. The remaining exact coefficients satisfy

$$\mu(\mathfrak{m})A_Y(\mathfrak{m}) = \sum_{\substack{\mathfrak{abc}=\mathfrak{m}\\ N\mathfrak{a},N\mathfrak{b}\le Y}} \mu(\mathfrak{c}), \qquad N\mathfrak{c} \ge D/Y^2 \ge D^{3/4}. \tag{101}$$

The small-short-divisor, long residual-Möbius cubic Gauss bilinear moment $\mathcal{L}_0[A_Y, A_Y]$ is still unestimated at the target.

23.6 The exact discrete top prefix, with every ray retained

There is also a direct reduction that does not introduce the principal-density split. Its exact support is

$$\mathcal{T}_X = \{(\mathfrak{w}, \mathfrak{s}) : N\mathfrak{w} < 64,\ \mathfrak{s} \text{ good},\ V < N\mathfrak{s} \le 2V,\ N\mathfrak{w}(N\mathfrak{s})^6 \le X\}. \tag{102}$$

There is no squarefree restriction on $\mathfrak{s}$. Overlap of $\mathfrak{w}, \mathfrak{s}$ and of $\mathfrak{k}, \mathfrak{s}$ remains allowed. Let $\widetilde{\Theta}$ be the same outer set as θ above, but with *every* original ρ, and put

$$\begin{aligned}\widetilde{d}_\theta[f_1, f_2] = \frac{A_1}{D} h_{AB\rho}(\mathcal{O}) c_{f_1}(\mathfrak{m})\overline{c_{f_2}(\mathfrak{n})}\frac{\epsilon(\eta)}{\sqrt{F}}\\ \cdot \frac{\mu(\mathfrak{e})\eta(\mathfrak{e})}{N\mathfrak{e}}\overline{\eta(\mathfrak{w})}.\end{aligned} \tag{103}$$

This is an explicit definition, not division by a density or a primitive value. With the original $\mathfrak{k}$ support as in (84), define the normalized exact piece

$$\begin{aligned}\mathcal{P}_X[f_1, f_2] = \sum_{\theta\in\widetilde{\Theta}} \widetilde{d}_\theta[f_1, f_2] \sum_{\mathfrak{k}} \frac{\mu(\mathfrak{k})\eta(\mathfrak{k})^2}{(N\mathfrak{k})^2}\\ \cdot \sum_{\mathfrak{s}:\ (\mathfrak{w},\mathfrak{s})\in\mathcal{T}_X} \bar{\rho}(\mathfrak{s})^6 \mathbf{1}_{(\mathfrak{s},\mathfrak{mn})=1} \dot{G}\left(\frac{\lambda_\theta(N\mathfrak{s})^6}{(N\mathfrak{k})^2}\right).\end{aligned} \tag{104}$$

Then $\mathcal{P}_X[A_U, A_U]$ is the actual original top primitive-prefix contribution divided by D. It is a signed representation-fixed piece, not a positive energy or the whole infinite dual sum.

Let $\mathcal{L}_X^K[f_1, f_2]$ be exactly (104) with $\dot{G}$ replaced by g_0. Only in that leading model can the finite $\mathfrak{k}$ sum be factored as $Q_2^{T_q}(K; \eta^2)$. The discrete $\mathfrak{s}$ sum and its endpoints remain. Unlike $\mathcal{L}_0$, this model has no $\kappa_S g(\mathfrak{m})g(\mathfrak{n})$ density, no $\rho^6 = 1$ projection and no integral replacing the $\mathfrak{s}$ family.

Proposition 35 (Direct original-prefix cutoff reduction)**.** *Put $\Lambda = A_1X/D^2$. With precisely the original support,*

$$|\mathcal{P}_X[A_U, A_U] - \mathcal{L}_X^K[A_U, A_U]| \ll B_W^2 (DX)^\epsilon A_1 V\Lambda, \tag{105}$$

and

$$\begin{aligned}|\mathcal{L}_X^K[A_U, A_U] - \mathcal{L}_X^K[A_Y, A_Y]| \ll B_W \mathcal{M}(W)(DX)^\epsilon A_1 V\\ \cdot \{Y^{-1/2} + D^{-1/6}\}.\end{aligned} \tag{106}$$

In particular the actual original piece satisfies

$$\begin{aligned}|\mathcal{P}_X[A_U, A_U] - \mathcal{L}_X^K[A_Y, A_Y]| \ll (DX)^\epsilon A_1 V\{B_W^2\Lambda\\ + B_W\mathcal{M}(W)(Y^{-1/2} + D^{-1/6})\}.\end{aligned} \tag{107}$$

The original unequal-norm projection is allowed.

At the critical pure parameters used above the three powers are again $-1/16$, $3/4$ and $31/48$, with exact zero coefficient tail when $Y = U$. This is a reduction of the *actual discrete* piece to a different, still unestimated, small-short-divisor model. That model retains every ρ, every nonunit S mode, the exact $\mathfrak{w}, \mathfrak{s}$ support, root/conductor data, original finite K and both long residual-Möbius column coefficients.

Separate open errors and scope. The accepted fixed-ray disk count gives the distinct top-band certificate

$$\frac{|\mathcal{P}_{\mathrm{top}} - M_{\mathrm{top}}|}{D} \ll B_W^2 (DX)^{\epsilon} A_1 X^{1/18} \{1 + \sqrt{A_1 X / D^2}\}. \tag{108}$$

Its pure critical power is $37/48 > 3/4$; this is an upper certificate, not a lower bound for the actual error. The direct reduction in Proposition 35 does not introduce that split: its discrete model retains those effects. No improved bound for the separate discrepancy, for the remaining small-cutoff models, or for the other original sectors is proved here. The result is an application of a named external cubic sieve to an actual arithmetic tail, not a new sieve, a complete analytic Lean formalization, a determinant/zero-location bridge or a proof of GRH. Fixed-profile constants and the explicit $\mathcal{M}(W)$ cannot be replaced by a uniform supnorm-only claim for varying profiles.

24 Ramified theta transforms and the original pure dual prefix

The exact discrete prefix introduced in Section 23 can be estimated with its original coefficient A_U. The argument constructs the previously missing finite-character transform at the ramified prime and groups all original short and Heath–Brown factors before taking a short-factor triangle. The resulting power is $101/144$ for this specified pure finite dual prefix. The whole signed comparison $13/12$, the global unsigned comparison $9/8$, and the remaining infinite-frequency assembly retain their earlier status.

The retained principal-density mean in the earlier discussion is a different expression with its own comparison problem. The result here uses every original ray component and the actual discrete sixth-part set. It supplies a new proof for the actual finite discrete prefix rather than modifying the historical cutoff stages.

24.1 The normalized object and its original support

Put $\omega = e^{2\pi i/3}$, $K = \mathbb{Q}(\omega)$, $\mathcal{O} = \mathbb{Z}[\omega]$ and $\lambda = 1 + 2\omega = i\sqrt{3}$. A good ideal avoids the original fixed set S, containing the primes above $2, 3$. All ideal columns are good and squarefree; different ideals of equal norm remain different columns. Use the unique multiplicative primary generator $\alpha_{\mathfrak{a}} \equiv 1 \pmod 3$. The original character ν has fixed finite order, with its conductor exclusions in S; no order-dividing-six hypothesis is imposed on ν.

Assume

$$D \geq 2^{64}, \quad 1 \leq U \leq \sqrt{D}, \quad A_1 \asymp H = D^{3/4}, \quad X = D^{3/8}, \quad V = X^{1/6}/2. \tag{109}$$

Only the original unweighted pure two-pivot block is asserted: $A_2 = 1$, its second good factor and the other four good factors are units, and u_0 is the fixed literal unit/S-supported factor with no good prime support. Keep its original capacity endpoints, nodal function G, and pre-S cutoff $K_{\mathrm{cut}} = \sqrt{b}$. An empty capacity interval gives zero. None of these data is recomputed during a coefficient decomposition.

Let W be one fixed possibly complex smooth function supported on $[1,2]$, with $W(1)=W(2)=0$, and write

$$\begin{aligned} B_W &= \|W\|_\infty, \qquad \widehat{W}(t)=\int_0^\infty W(x)x^{-it}\,\frac{dx}{x}, \\ \mathcal{M}(W) &= \frac{1}{2\pi}\int_{\mathbb{R}}|\widehat{W}(t)|\,dt, \qquad\qquad V_1(W)=B_W+\int_1^2|W'(x)|\,dx. \end{aligned} \tag{110}$$

These are seminorms of the common column profile. No uniform derivative hypothesis on the sharp capacity interpolant is added. The exact frequency set is

$$\mathcal{T}_X=\left\{(\mathfrak{w},\mathfrak{s}):\ \begin{array}{l}\mathfrak{w}\text{ integral, }N\mathfrak{w}<64,\quad \mathfrak{s}\text{ good integral,}\\ V<N\mathfrak{s}\le 2V,\quad N\mathfrak{w}(N\mathfrak{s})^6\le X\end{array}\right\}. \tag{111}$$

There is no squarefreeness requirement on $\mathfrak{s}$. The permitted overlaps among $\mathfrak{k},\mathfrak{s},\mathfrak{w}$ remain permitted, including nonunit S-supported $\mathfrak{w}$.

Here is a full definition of the piece being bounded. Let $\Omega=(A,B,\rho,\mathfrak{m},\mathfrak{n})$ range over the original finite source/ray components and their ordered coprime good squarefree columns in $[D,2D]$, with the original source classes. An original unequal-norm projection may be imposed consistently. Use the actual primitive character η_Ω of (61), its actual S-conductor $\mathfrak{f}_S$, and

$$F_\Omega=N\mathfrak{f}_S\,N\mathfrak{m}\,N\mathfrak{n},\qquad \mathfrak{q}=\mathfrak{m}\mathfrak{n},\qquad c_U(\mathfrak{m})=A_U(\mathfrak{m})\nu(\mathfrak{m})R_{\mathfrak{m}}(u_0)W(N\mathfrak{m}/D).$$

The numerator value $R_{\mathfrak{m}}(u_0)$ includes its original raw zeros. Put $P_S=\prod_{\mathfrak{p}\in S}\mathfrak{p}$. With $C_0=3/(4\pi^2)$, the normalized prefix is

$$\begin{aligned} \mathcal{P}_X[A_U,A_U]=\frac{A_1}{D}\sum_{\Omega}h_{AB\rho}(\mathcal{O})c_U(\mathfrak{m})\overline{c_U(\mathfrak{n})}\frac{\varepsilon(\eta_\Omega)}{\sqrt{F_\Omega}} \\ \cdot\sum_{\mathfrak{e}\mid P_S}\frac{\mu(\mathfrak{e})\eta_\Omega(\mathfrak{e})}{N\mathfrak{e}}\sum_{\substack{\mathfrak{k}\text{ good,SF}\\ N\mathfrak{k}\le K_{\rm cut}}}\frac{\mu(\mathfrak{k})\eta_\Omega(\mathfrak{k})^2}{(N\mathfrak{k})^2} \\ \cdot\sum_{(\mathfrak{w},\mathfrak{s})\in\mathcal{T}_X}\overline{\eta_\Omega(\mathfrak{w})}\,\overline{\rho(\mathfrak{s})}^{\,6}\mathbf{1}_{(\mathfrak{s},\mathfrak{q})=1}\dot{G}\left(\frac{A_1N\mathfrak{w}(N\mathfrak{s})^6}{(N\mathfrak{k})^2N\mathfrak{e}F_\Omega}\right). \end{aligned} \tag{112}$$

All sums are finite. The ray label ρ here is distinguished from the Jacobi column weight below. Values at $\mathfrak{e}$ and $\mathfrak{w}$ are separate primitive values, including zero; a conductor drop does not restore a deleted prime. The original $1/D$ occurs once. The raw numerator is $D\mathcal{P}_X$.

Define $\mathcal{L}_X^{K_{\rm cut}}$ by replacing $\dot{G}$ in (112) by $g_0=C_0^{-1/2}\int_0^\infty G(v)\,dv$, with every other datum fixed. For the original coefficients the established real-axis estimate is

$$|\mathcal{P}_X-\mathcal{L}_X^{K_{\rm cut}}|\ll B_W^2(DX)^{\epsilon}A_1V\Lambda,\qquad \Lambda=A_1X/D^2. \tag{113}$$

It uses $0\le G\le 1$, its fixed compact support and a pointwise Bessel remainder. The value g_0 may vanish and is never divided out.

24.2 Actual finite characters and a sharp ramified row

The actual root calculation leaves one angular Jacobi gauge and a cubic Gauss root in each varying column. On good squarefree ideals the Jacobi function is $\rho_J=\mu\vartheta$, where $\vartheta=\Psi/\sqrt{N}$ has infinity type $(1,0)$; here $\Psi(\mathfrak{a})$ is the chosen generator 1 mod 2λ. The arithmetic μ belongs in the coefficient

opening. There is exactly one varying ϑ, or its conjugate; an additional finite-order assertion about ρ_J would be false.

Write $\chi_{3,\mathfrak{a}}$ for the positive cubic residue character, with its zero at every shared prime. The source Gauss root is

$$G_{\mathfrak{a}}^{\mathrm{src}} = \frac{1}{\sqrt{N\mathfrak{a}}} \sum_{x \bmod \alpha_{\mathfrak{a}}} \chi_{3,\mathfrak{a}}(x) e^{2\pi i \operatorname{Tr}(x/\alpha_{\mathfrak{a}})}. \tag{114}$$

The actual root uses $\lambda\alpha_{\mathfrak{a}}$ in this additive denominator. The normalization of the angular factor is explicit:

$$\begin{aligned} \Theta_{\ell}(\mathfrak{a}) &= (\alpha_{\mathfrak{a}}/|\alpha_{\mathfrak{a}}|)^{\ell}, \\ \vartheta(\mathfrak{a}) &= \Theta_1(\mathfrak{a})\overline{\chi_{3,\mathfrak{a}}(2)}, \\ G_{\mathfrak{a}}^{\mathrm{act}} &= \chi_{3,\mathfrak{a}}(\lambda) G_{\mathfrak{a}}^{\mathrm{src}}. \end{aligned} \tag{115}$$

Thus the two fixed corrections may be written as source frequency $12t$. The factor 12 belongs to the fixed finite character data; $12t$ is not declared a good squarefree moving level.

After exact product-character expansion of the joint finite S/source/Γ selectors, every remaining own-variable phase has the form $\Theta_{\pm 1}\kappa$, where κ belongs to a fixed finite residue family, with its original modulus-zero extension. It includes higher 3-primary conductors of ν. The finite source quotient is a fixed ray quotient; fixed numerator symbols are finite Kummer characters. The full Fourier sums retain their actual coefficient masses and reconstruct the primitive $\mathfrak{e}/\mathfrak{w}$ zeros. Growing other-column deletions remain literal masks.

For a physical good squarefree frequency $\mathfrak{t}$, an exact interval $I \subset [T, 2T]$, and a finite good deletion ideal Q, write the actual row in either required conjugate orientation as

$$\mathcal{F}_{\psi}(T; \mathfrak{t}; Q, I) = \sum_{\substack{\mathfrak{a} \text{ good,SF} \\ N\mathfrak{a} \in I,\ (\mathfrak{a},Q)=1}} \psi(\mathfrak{a}) \overline{G_{\mathfrak{a}}^{\mathrm{act}}} \chi_{3,\mathfrak{a}}(\alpha_{\mathfrak{t}}) \frac{W(N\mathfrak{a}/T)}{\sqrt{N\mathfrak{a}}}. \tag{116}$$

Here ψ is from this actual fixed family, including its angular factor. It is not an arbitrary bounded multiplicative function.

Theorem 36 (The actual fixed-family sharp row)**.** *For every such member, both conjugate orientations and every physical good squarefree* $\mathfrak{t}$*, including the unit,*

$$|\mathcal{F}_{\psi}(T; \mathfrak{t}; 1, I)| \ll_{\mathrm{fixed},\epsilon} V_1(W)(2 + TN\mathfrak{t})^{\epsilon} \min\{\sqrt{T},\ T^{1/4}(N\mathfrak{t})^{3/8}\}, \qquad T \geq 1. \tag{117}$$

The constant may depend on the fixed conductor/character family. A literal unit calculation extends the estimate to $1/2 \leq T < 1$*; for* $T \leq 1/2$ *the row is zero with the stated endpoint convention.*

Proof. Use the unconditional theta modularity and known 27 integral cusps in Dunn–Radziwiłł, arXiv:2109.07463v3, Section 5 and Appendix A. Its weight-zero cubic theta θ is automorphic with Kubota multiplier χ on $\Gamma_2 = \mathrm{SL}_2(\mathbb{Z})\Gamma_1(3)$, where $\Gamma_1(3)$ is the principal congruence subgroup of $\mathrm{SL}_2(\mathcal{O})$. Fix its known representatives γ_j and $F_j = \theta \circ \gamma_j$. The correct ramified coefficient projector acts on $\lambda^3\mu$. Its primary selector is $\mathbf{1}_{\lambda^3\mu \equiv 1 \bmod 3}$. A finite Fourier expansion writes it, and the away-3 periodic twist, as actual rational translates of theta.

For $a = A/C$ reduced, choose $\gamma = \left(\begin{smallmatrix} A & B \\ C & D_0 \end{smallmatrix}\right)$ of determinant one, and choose j so that $\delta = \gamma\gamma_j^{-1} \in \Gamma_2$. Use $U(x) = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ and $W_C = \left(\begin{smallmatrix} 0 & -1/C \\ C & 0 \end{smallmatrix}\right)$. Then

$$\gamma^{-1}U(a) = U(-D_0/C)W_{-C}, \qquad \theta(U(a)w) = \chi(\delta)F_j(U(-D_0/C)W_{-C}w).$$

The multiplier is its literal Kubota value. Pure nonzero angular derivatives kill the pulled-back radial constants. Their Mellin identity has factor $(NC)^{1-2s}$, including the Jacobian. The explicit cusp coefficients have absolute Dirichlet norm at most

$$3^8\zeta_K^{(\lambda)}(\sigma)\zeta_K^{(\lambda)}(3\sigma-\tfrac{1}{2}),\qquad 1<\sigma<13/12.$$

Normalized finite Fourier mass is $O_{\mathrm{fixed}}(\sqrt{N\mathfrak{t}})$ and each reduced denominator has norm $O_{\mathrm{fixed}}(N\mathfrak{t})$. This gives the conservative dual cost $(N\mathfrak{t})^{3/2+\epsilon}$.

Coefficientwise cube completion retains the raw $L(3s-\frac{1}{2},\kappa^3\Theta_{3\ell})$, with every stipulated Euler omission. Apply convexity to this entire completed numerator first. Its reciprocal is used only where $\Re(3s-\frac{1}{2})>1$, by absolutely convergent Euler products. Exact half-integer Perron thresholds and partial summation give (117). The optimized height is $(T/(N\mathfrak{t})^{3/2})^{1/4}$ when this is at least one; the elementary count gives the other branch. The extended edition retains the constants and the zeroth-angular residues explicitly. □

24.3 The full original coefficient and its finite prefix

On the actual contributing annulus $N\mathfrak{m}>U$, put

$$B_U(\mathfrak{m}):=\mu(\mathfrak{m})A_U(\mathfrak{m})=\sum_{\substack{\mathfrak{abc}=\mathfrak{m}\\ N\mathfrak{a},N\mathfrak{b}\le U}}\mu(\mathfrak{c}).\tag{118}$$

This B_U is not the project's original C_U. Equation (118) is not asserted for arbitrary small columns or the unit. Unit factors inside an annular factorization remain present.

Theorem 37 (Original pure finite dual-prefix bound)**.** *Under* (109)*, with the complete original support and fixed data of* (112)*,*

$$\begin{aligned}|\mathcal{P}_X[A_U,A_U]|\ll_{\mathrm{fixed},\epsilon}(DXA_1)^{\epsilon}A_1V\{&B_W[\mathcal{M}(W)+V_1(W)]D^{-1/9}\\&+B_W^2[\Lambda+D^{-1}]\}.\end{aligned}\tag{119}$$

In particular its normalized leading power is

$$13/16-1/9=101/144<3/4,\qquad 3/4-101/144=7/144.\tag{120}$$

The raw numerator has one additional factor D.

Proof. Open (118) and the exact $J=5$ finite Heath–Brown identity at $z_0=(2D)^{1/5}$. There are at most eleven factors, including the original $\mathfrak{a},\mathfrak{b}$. Choose their grouping from fixed independent dyadic scales. A balanced whole-factor partition has a physical cubic matrix bound $\mathcal{M}(W)D^{1/3+\epsilon}$, using the pinned de Faveri, arXiv:2610.04045v1, Theorem 1.1 and Example 2.2, at order three.

An unbalanced box has one long unrestricted coefficient-one factor $\mathfrak{v}$. At $D\ge 2^{64}$ neither an original short factor nor a short Möbius factor can occupy that scale. The product $\mathfrak{p}$ of all other factors has norm $O(D^{1/3})$ and its intrinsic signed assignment coefficient is divisor-bounded. This retains the $j=1$ boxes, which are all balanced.

For $N\mathfrak{p}>Z$ the cubic matrix gives $\mathcal{M}(W)D^{\epsilon}[D^{1/3}+\sqrt{D/Z}]$. For $N\mathfrak{p}\le Z$, freeze the original opposite column $\mathfrak{n}$ first. Keep $\mathbf{1}_{(\mathfrak{p},\mathfrak{n})=1}$ and all prefix $\mathfrak{k}/\mathfrak{s}/\mathfrak{w}$ zeros in its actual coefficient. The remaining long-variable deletion is $Q=\mathrm{rad}_{\mathrm{good}}(\mathfrak{pnksw})$. Its exact divisor $\mathfrak{h}$ changes frequency to $\mathfrak{pkh}$ and length to $D/(N\mathfrak{p}N\mathfrak{h})$. Including the original prefix weight, Theorem 36 gives

$$\frac{1}{\sqrt{N\mathfrak{p}N\mathfrak{h}}}\left(\frac{D}{N\mathfrak{p}N\mathfrak{h}}\right)^{1/4}(N(\mathfrak{pkh}))^{3/8}=D^{1/4}(N\mathfrak{k})^{3/8}(N\mathfrak{p})^{-3/8}(N\mathfrak{h})^{-3/8}.\tag{121}$$

Thus the small-prefix column costs $V_1(W)D^{\epsilon}D^{1/4}(N\mathfrak{k})^{3/8+\epsilon}Z^{5/8}$. Choose $Z = D^{2/9}$; both column costs have power $7/18$.

Only after that bound take the opposite-column elementary $B_W D^{1/2+\epsilon}$ count. Use the original A_1/D once, sum the original finite $\mathfrak{k}$ with the convergent positive majorant $\sum(N\mathfrak{k})^{-13/8+\epsilon}$, and count the exact $\mathfrak{s}$ family by $O(V)$. Optional unequal norms cost the same fixed-norm fibre bound $O(B_W^2 A_1 V D^{-1+\epsilon})$. Finally (113) applies directly to A_U, A_U. No A_U-to-A_Y replacement or common-long-divisor opening is used. □

The estimate has fixed-conductor and explicit profile-seminorm dependence. It does not supply a supremum-norm estimate for varying W_D, an arbitrary weighted-row operator, the infinite dual complement, mixed or neighboring blocks, short/principal assembly, an independent positive determinant, or a zero-location theorem. The Dirichlet, Hecke, Dedekind, elliptic and general Artin obligations remain separate. No analytic/category Lean formalization or GRH proof is claimed.

25 Paired cutoffs, exact divisor signs and proved regimes

The comparison in this section concerns a specified part of the signed coprime-column form for the original unweighted, nonprincipal, two-pivot Eisenstein rows. We use the physical ideal norms and the original ordered cutoff. Let D, U, H be the column, cutoff and capacity parameters, with $U^2 \le D \le 7U^2/2$ and $A_1 A_2^2 \ll H \le D$. The first two row ideals have dyadic norms A_1, A_2, $R = A_1 A_2$, and the actual primitive conductor of row u has norm $Q_u \asymp R$. The finite bad-prime set S, row type, and twist are fixed. The same smooth complex profile W supported on $[1, 2]$ is used on both sides. Ideal labels, including conjugate primes of the same norm, are retained individually.

25.1 The selected pairing

For each good squarefree intermediate Möbius divisor d, put $E = Nd$. Let V_d be its original row amplitude and $\mathcal{E}_d$ the equal-norm part of the original middle Poisson window. We study selections of

$$\frac{2}{D}\operatorname{Re}\sum_d \mu(d)\langle \mathcal{E}_d, V_d - \mathcal{E}_d\rangle. \tag{122}$$

Here the counting inner product is linear in the first argument. This is one term in the expansion of the original signed norm. In particular the intermediate d is different from the gcd of the final column pair.

On an original allocation let $z = e_0 z_0$ with $J \le Nz < 2J$, let $e_1 \mid \operatorname{rad}(Sd)$ be the other deletion label, and put $h = re_0$, with r the original squarefree short divisor. Set

$$X = \frac{D}{E(Nr)^2}, \qquad X' = \frac{X}{(Ne_0)^2}, \qquad M = \frac{R\,Ne_0\,Ne_1\,J}{X}.$$

The original middle window has $X' > B_*$ and $K_0 < Nk \le MD^{\eta}$, for its specified B_*, K_0, η. The primal test discards the guaranteed zero cells $X/((Ne_0)(Ne_1)J) \le 1/2$, and the original weights retain any further zeros. An allocation with $X' \le B_*$ belongs to the direct component. Its re-expressed Poisson sum stays in the complement in (122).

On the equality side write $z_{0,0} = g\ell$ and $k_0 = g\bar{\ell}$, so $Nz_{0,0} = Nk_0$. On the partner side the selected changes in the good exponents consist of erasing the squarefree cutoff part x, erasing a part of the good conjugate overlap, and inserting allowed positive increments at other good primes outside d. A conjugate erasure removes the full original exponent, which can be two. At an insertion prime the old residue plus the increment is at most five. Changes at the good primes of d keep their

exact new residues. For fixed other labels write $z_{0,1} = z_* x$ and $k = Ik_*$. The physical insertion ideal I keeps each exponent $1, \dots, 5$ and is disjoint from the selected erasure supports and d. Thus $Nx \asymp B$ and $NI \asymp K$ are physical lengths. The two varying raw phases are $\psi_u(x)\overline{\psi_u(I)}$, or their conjugates if the pairing is conjugated. The fixed row factors retain the union of deleted primes, the literal section, the twist and the primitive S-values.

The subtraction on the partner imposes $Nz_* Nx = Nk_* NI$ only in its original middle window. At fixed x this fixes at most one integer norm for I, and the reverse count fixes at most one norm for x. An integer norm supports at most divisor-many ideals on the polynomial clip. With the physical inverse $1/Nx$, the paired count therefore costs $\min(1, K/B)$ up to subpowers. The two half radial powers leave

$$(B/K)^{1/2} \min(1, K/B) = \sqrt{\min(B/K, K/B)}.$$

The remaining harmonic weights give $O(R/E)$ for this subtraction at each d, with the inherited profile and subpower costs. Both original endpoints remain in the count. Equal norms in the direct and other components remain in the pairing.

25.2 Target bounds on the selected cross

The bounds have the common cost $O_{\epsilon,S,\nu,u_0}((DH)^{\epsilon} \mathcal{M}_{24}(W)^2)$, where $\mathcal{M}_{24}$ controls the indicated smooth profile derivatives. The live partner primal test supplies $E \ll D/B$ and partner radial scale $y_1' \gg Nk/R \gg K/R$. The proof first sums each actual column array. It then uses the row comparisons and the original label sums.

At the mixed scales $A_1 \asymp D^{5/12}$, $A_2 \asymp D^{1/6}$, the selected normalized cross has upper power $57/80$ over all physical ratios and all d. If $H \ll D^{3/4}$, a perturbed primitive sharp moment gives the target power $3/4$ for the same selected cross on

$$\frac{7}{24} \le \log_D A_1 \le \frac{439}{912}, \qquad A_1 A_2^2 \ll H.$$

The second pivot need only satisfy the stated original admissibility and capacity. The conductor comparison uses the actual R.

At the pure critical scales $A_1 \asymp R \asymp D^{3/4}$, $A_2 \asymp 1$, the target power $3/4$ holds over all d in each of the following physical sectors:

- $K \gtrsim D^{1089/1420}$, with any erasure length B.
- $B \gtrsim D^{1355/1482}$, with any insertion length K.
- $B \gtrsim R$ and $K \lesssim D^{921/1652}$.

The flexible central windows give further short and long threshold curves. There are also separate comparisons for a small fixed residual label product and for a small d-prefix. The comparable pure all-divisor comparison has power $55/68$, and the target holds on $Nd \le D^{9/68}$ in that sector. Each assertion concerns the selected changes described above. The remaining pure rectangles and the coherently averaged signed divisor sum are open.

25.3 Ordered assignment signs

Fix an ordered factorization $\zeta = \zeta_1 \zeta_2$ of the actual squarefree cutoff label $\zeta = e_0 z_0$ on either side, with $(d, \zeta) = 1$ and $N\zeta_i \le U$. Write $T_i = U/N\zeta_i$. The assignment sum multiplying $\mu(\zeta)$ is

$$F_d = \sum_{\substack{a,b \mid d,\ (a,b)=1 \\ Na \le T_1,\ Nb \le T_2}} \mu(a)\mu(b).$$

Complete the b-sum and then subtract its excluded terms. With $m = d/b$ in the subtraction, squarefreeness gives the exact identity

$$\frac{F_d}{\mu(d)} = \mathbf{1}_{E\le T_1} + \mathbf{1}_{E\le T_2} - 1 - \sum_{\substack{m\mid d\\ T_1<Nm<E/T_2}} \sum_{\substack{a\mid m\\ Na\le T_1}} \mu(m/a). \tag{123}$$

The complete b-sum forces $a = d$. The unit $m = 1$ gives the indicator at T_2, while the nonunit terms with $Nm \le T_1$ vanish by the full inner Möbius sum. Both inequalities of the remaining band are strict. Below the product cap the band is empty. The three possible assignment values are $\mu(d), 0, -\mu(d)$ according as both, just one, or neither individual capacity reaches E.

Fix the row, both ordered factorizations and the primal and dual labels. Select both e_1 labels to be fixed S-supported ideals, take the good changed-residue ideal on d to be the unit, and require d to avoid the full fixed prime union of the row and the primal and dual labels. When $EN\zeta \le U^2$ on both sides, their capacity step functions multiply the original outer Möbius sign. The fixed-field deleted Möbius estimate supplies subpower cancellation for the resulting fixed-label scalar on $Nd \asymp Y$ with $Y \ge D^\kappa$ for a fixed $\kappa > 0$. In the pure comparable block, a physical Nx-interval of width $O(BD^{-1/8})$ gives the target after the full divisor sum by a separate sparse erased-column estimate. This includes the narrow interval immediately above the product cap on the actual partner allocation. The wide correction and the remaining coherent signed estimate require additional arguments. The whole signed and unsigned powers 13/12 and 9/8, the independent determinant comparison, and the scope-specific zero-location obligations remain as in the preceding sections.

26 Wide cutoff corrections and full-frequency compensation

Keep the pairing (122), the exact ordered cutoff identity (123), and all the original allocation, primitive-conductor and raw-zero conditions. At one live allocation put $h = re_0$, $E_1 = Ne_1$, and write the actual cutoff ideal as $\zeta = e_0 z_0 = \zeta_1\zeta_2$. Thus $T_i = U/N\zeta_i \ge 1$, $E = Nd$, and

$$\Omega = \frac{E}{T_1T_2} = \frac{EN\zeta}{U^2} < \frac{14}{E_1Nh}.$$

Let $\mathcal{P}$ consist of the good prime ideals of norms seven or thirteen, with the original S-deletion. These are at most four separate ideal labels. Put $s(E) = \mathbf{1}_{E\le T_1} + \mathbf{1}_{E\le T_2} - 1$. The exact wide correction is

$$\begin{aligned} F_d(T_1,T_2) &= \mu(d)\big(s(E) + C_d(T_1,T_2)\big),\\ C_d(T_1,T_2) &= \sum_{\substack{mn=d\\ Nm>T_1,\ Nn>T_2}} \left(1 - \sum_{\substack{p\in\mathcal{P}\\ p\mid m}} \mathbf{1}_{Nm>T_1Np}\right). \end{aligned} \tag{124}$$

The two factors are coprime because d is squarefree. The strict tests retain the original cutoff endpoints. For $E_1Nh \ge 2$ only the unweighted factorization count remains. For $E_1Nh \ge 14$ the correction vanishes.

At fixed row and fixed primal and dual labels, prescribe the exact intersection δ of d with the full forbidden prime union, including $\mathcal{P}$. In a divisor dyad $Y \le Nd < 2Y$ put $Y' = Y/N\delta$. For fixed $\kappa > 0$ and $Y' \ge D^\kappa$, the fixed-label scalar after taking out its fixed coefficient and row phase has bound

$$\ll_\kappa \mathcal{M}_{24}(W)^2 \frac{4^{\omega(\delta)}\log^C(2D)}{Y\,N\delta}\,\exp\big(-c_\kappa\sqrt{\log(2Y')}\big). \tag{125}$$

The estimate includes both wide corrections, with the original full prime deletions. It gives subpower cancellation before a coherent average over the rows and other labels.

For each $p \in \mathcal{P}$, replacing the partner coefficient by its signed p-subtraction in (124) gives a zero primal amplitude $V_d^{[p]}$. The exact full-frequency complement is $V_d^{[p]} - \mathcal{E}_d^{[p]} = -\mathcal{E}_d^{[p]}$, where $\mathcal{E}_d^{[p]}$ is its middle equality projection. The original elementary equality norm gives a contribution of order at most $R(DH)^{\epsilon}\mathcal{M}_{24}(W)^2$ after the full divisor sum. This comparison requires all partner modes, including the re-expressed direct component.

In the pure comparable sector $A_1 \asymp R \asymp D^{3/4}$, $A_2 \asymp 1$, $B, K \asymp R$, the equality-side wide correction reaches $D^{3/4+\epsilon}$ on $Nd \le D^{13/56}$ with the inherited profile cost. Its partner can carry the original coefficient, the step part, or the correction part. Thus this prefix includes their double correction. The remaining coherent wide sum and the other signed components remain open.

27 Joint moments for the wide cutoff pairing

We work at the pure scales $A = A_1 \asymp R \asymp D^{3/4}$, $A_2 \asymp 1$, with $U^2 \le D \le 7U^2/2$. The rows are the original unweighted nonprincipal rows, with fixed S, ν, u_0, their literal ray and capacity restrictions, and their primitive conductors $Q_u \asymp R$. Use one smooth complex W supported on $[1, 2]$, the full ordered cutoff, and the original raw zeros. A target bound means $O((DH)^{\epsilon}\mathcal{M}_{24}(W)^2 D^{3/4})$ for the specified contribution to the normalized pairing (122). Constants may depend on the fixed arithmetic data, scale constants and $\epsilon > 0$. Put $E = Nd$ and $e = \log_D E$. Here d is the intermediate divisor from the exact Möbius identity. All estimates include the original middle equality subtraction.

Theorem 38 (Full-frequency partner unit part)**.** *Take the unit part of C_d on the partner side of* (124)*, with all partner frequencies including the re-expressed direct term. A selection of the actual ordered pairs $mn = d$ may depend on d, m, n. Write $L_1 = U/Nm$, $L_2 = U/Nn$ and $l_s = \log_D \max(1, \min(L_1, L_2))$. The target holds over all $E \lesssim D^{1/6}$, and more generally on the dyads with $e \le 1/4$ and $l_s \ge 1/4 + e/2$ at the level of powers. The remaining upper cost at a single dyad with $e \le 1/4$, relative to R/E, has exponent $(1/4 + e/2 - l_s)_+/6$, where $x_+ = \max(0, x)$. The all-divisor bound for this component has power* $37/48$*.*

The low-divisor estimate uses the whole wide product support, even for balanced pairs with bounded nonunit d. Its gain comes from the forced radical of the actual product ideal in the joint positive norm. For instance, at $e = 1/4$ with the longer length of order $D^{1/2}$, the upper comparison still costs $D^{1/48}$ above R/E. This excess describes the estimate.

Theorem 39 (Selected partner frequencies)**.** *Let $T \le N\mathfrak{k} < 2T$ be an actual partner frequency dyad and write $t = \log_D T$. A selector of modulus at most one may depend on $\mathfrak{k}$, the actual equality allocation and the fixed partner labels. It is independent of the row and the two varying partner cutoff ideals. In the actual Poisson representation of the unit part, including the direct term and the corresponding selected middle subtraction, the target holds in the following regions:*

$$\begin{array}{c|l} 0 \le e \le 1/8 & t \le 15/32 + 3e/8 \\ 1/8 \le e \le 1/4 & t \le 33/64 \\ 1/4 \le e \le 5/16 & t \le 17/64 + e \\ 5/16 \le e < 27/56 & t \le 11/32 + 3e/4 \\ e \ge 27/56 & \textit{all frequency lengths.} \end{array} \tag{126}$$

Thus $T \lesssim D^{15/32}$ is sufficient over all intermediate divisors. Constant factors at the endpoints are allowed.

For selectors independent of the varying equality labels, further regions follow by grouping frequencies with prescribed ideal parts. If the rational content of $\mathfrak{k}$ is at least D^b, with $0 \le b \le 9/28$, then $T \lesssim D^{9/20+3b/5}$ suffices across all d. Here the content is the largest positive integer prime to the rational primes under S whose principal ideal divides $\mathfrak{k}$. At $E \gtrsim D^{1/4}$, content of order at least $D^{41/144}$ permits $T \lesssim D^{2/3}$. A good sixth-power base of norm at least of order $D^{1/16}$, or an S-part of norm at least of order $D^{3/10}$, suffices at every frequency length. These are conditions on the actual frequency, with all ideal labels and raw zeros retained.

Theorem 40 (Equality-wide correction)**.** *In the equality-side wide correction, write the equal-norm pair as $z_0 = g\ell,\ k = g\bar{\ell}$. Let ℓ_{g} retain the prime ideals of ℓ whose conjugates also avoid S. Against the full original partner complement, or its full-frequency step or wide part with its own subtraction, the target holds on $N\ell_{\mathrm{g}} \gtrsim D^{\theta(e)}$, where*

$$\theta(e) = \begin{cases} -17/12 + 37e/6, & 17/74 \le e \le 1/4, \\ 11/24 - 4e/3, & 1/4 \le e \le 11/32, \\ 0, & \textit{otherwise.} \end{cases}$$

In the distinct pure comparable selected rectangles $B, K \asymp R$ of the preceding wide analysis, the equality-side prefix extends to $Nd \lesssim D^{111/476}$, with the original allocation tests and the same choices of partner coefficient.

The small-conjugate-support band, unbalanced middle divisors and small-part main frequency range leave contributions to the coherent signed problem. The full signed assembly and the independent determinant and zero-location arguments remain open.

Classical reference NIST Digital Library of Mathematical Functions, Section25.15, Dirichlet L-functions, equations 25.15.5–6; primitive functional equation and finite Gauss sums (snapshot pinned 4 November 2026). The normalization and explicit finite Gauss sums used here are stated above.