Infinite dessin orders, projective resolutions and spectral determinants

Constructions, invariant theory and the arithmetic comparison problem

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Abstract

We work throughout with infinite-dimensional locally gentle algebras and completed surface orders arising from explicit cellular clean dessins on compact Riemann surfaces. We construct their projective resolutions, Cartan determinants and dual dessins; develop determinant semi-invariants, Schur modules and cyclic-quiver character kernels; and calculate actual lattice almost-split sequences and an affine-type diamond quotient. A constrained-path Fock algebra retains the gentle relations and has a compact extension with incidence boundary map. Its degree spectral triple and an inserted arithmetic Fredholm family are explicit. Jacobi sums provide theta-dessin cellular Frobenius determinants with a positive local metric; inert primes obstruct the same real two-dimensional carrier. Recent spectral approaches are compared with their actual hypotheses. Selected power-series, module and matrix statements are verified in Lean4. A derived branch functor repairs projective determinant multiplicity; tame descent recovers a hereditary order but forces multiplicities for full-deck coefficient kernels. We also give a requested-error adaptive theta/readout construction for three fixed completed Dirichlet functions, with executed rational training and explicit sampling budgets, and derive a cutoff-uniform original-family type-I remainder envelope, retaining its unresolved exponent loss. A classical quadratic pivot refines the cube-factor sector, and a joint Gauss identity preserves conductor phases. An ordinary all-character conductor comparison improves the unsigned raw envelope to $H+DH^{1/6}+(DH)^{2/3}$, while an executed finite arithmetic Fourier readout certifies correlation and signed-observable errors without asserting a varying-family signed saving. A moving common-profile construction aligns the actual coefficients on nonprincipal sixth-power rows, and an exact joint example rules out variance-only centering; the fixed-profile signed target remains open. An exact conductor-12 character twist and gcd–Möbius inversion give a separate bound for the signed coprime slice, with the former 7/6 comparison retained as history. A signed common-divisor zeta expansion and a complete-column exact laboratory retain every raw mask; weighted divisor certificates do not close the required asymptotic saving. An exact rough-prime support identity gives an explicit target-budget mask correction; the remaining coherent signed main term is still unestimated. A pinned fixed-order preprint theorem, applied to the physical sixth-power-free radicands, improves the global unsigned critical power to 9/8 and the $A_6 = 1$ signed slice to 13/12. Mean-retaining coherent reductions and conditional scalar-input diagnostics keep the fixed-profile signed target open. The arithmetic positive determinant comparison remains open; no proof of GRH is claimed. Primitive root normalization, all-component finite dual prefixes, actual-cutoff Schur transfer and a full equal-norm localization retain the coherent unequal-norm target as an open problem. An actual Gauss-weighted short-divisor tail admits a physical cubic-matrix estimate. A ramified theta construction now covers the actual fixed finite-character family, and a whole-factor partition bounds the original pure finite dual prefix with normalized power 101/144, retaining every ray, raw zero and finite cutoff. The infinite-frequency complement and full signed assembly remain open; the whole signed and unsigned exponents are unchanged. The paired cutoff analysis gives target-size estimates on stated pure and mixed selections of the signed equality–complement cross, with an exact ordered divisor identity.

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1 Cellular dessins and their infinite orders

We start with the surface-order constructions of Schreiber [1, 2], retaining the original sources unchanged. The generic-resolution source [3] is background, not a substitute for exactness of a particular complex. All primary algebra examples below come from explicit clean Belyi maps. Finite-dimensional objects occur as modules or specified quotient strata of those infinite algebras, not as replacements for them. The initial arithmetic target is a primitive nonprincipal Dirichlet L-function. Finite-order Hecke characters, Dedekind zeta functions and general Artin representations require separate comparison theorems; Artin holomorphy is not assumed.

A clean dessin on a compact oriented surface is specified by a nonempty finite dart set H, a fixed-point-free involution α, and a permutation σ such that $\langle \alpha, \sigma \rangle$ is transitive. Contract the white vertices of degree two. The resulting cellular graph has vertices the σ-orbits, edges the α-orbits, and faces the ϕ-orbits, where

$$\tau = \sigma^{-1}, \qquad \phi = \alpha\sigma^{-1}, \qquad 2 - 2g = \#\operatorname{cyc}(\sigma) - |H|/2 + \#\operatorname{cyc}(\phi).$$

Write $[h] = \{h, \alpha h\}$. Our quiver has vertices these edges and arrows $a_h : [h] \to [\tau h]$. Paths are composed in travel order: ab traverses a first. Set

$$A = kQ/(a_h a_{\phi h} : h \in H), \quad J = A_{>0}, \quad \Lambda = \varprojlim A/J^m.$$

Each vertex has two incoming and two outgoing arrows. Each arrow has precisely one permitted successor $a_{\tau h}$ and one forbidden successor $a_{\phi h}$; these are distinct because α has no fixed points. The same holds for predecessors. This is the *locally gentle* condition. Every permitted cycle repeats indefinitely, so A is infinite dimensional. We reserve "gentle" without qualification for the convention additionally imposing finite dimension.

Proposition 1.1 (The actual split order)**.** *For a permitted cycle c of length ℓ, let B_c be the hereditary cycle order in $M_\ell(k[[u_c]])$: diagonal and forward matrix entries are in $k[[u_c]]$, and entries crossing the chosen cycle cut are in $u_c k[[u_c]]$. Then Λ is the suborder of $B = \prod_c B_c$ whose two diagonal constant terms labelled $h, \alpha h$ agree. With $R = k[[u]]$ acting by $u \mapsto (u_c I)_c$, it is finite free of rank $\sum_c \ell(c)^2$; the conductor contains uB.*

Proof. In a cycle order a path of length less than ℓ is the corresponding matrix unit, multiplied by u_c exactly when it crosses the cut; another full turn multiplies it by u_c. Thus completed paths in each permitted branch give all entries of B_c. An arrow belongs to only that branch, whereas $e_{[h]}$ has diagonal entries at both occurrences $h, \alpha h$. Hence the only condition on branchwise

formal coefficients is the displayed equality of diagonal constants. This proves both injectivity and surjectivity of the path-normal-form identification. Multiplication respects it: switching branches across a paired edge is precisely a forbidden pair. The quotient B/Λ is killed by u, so the submodule has the same generic rank as B. It is torsion free and finite over the DVR R, hence free of that rank. The path and u-adic topologies agree: $uA \subset J$, and if L is the longest permitted cycle, any path of length at least $L+1$ is u times a shorter nonempty path. Thus $J^{L+1} \subset uA$. This also justifies the completion used in the identification. □

The path-degree variable q below is not u: one full turn has path degree $\ell(c)$ but u-degree one. Formal, Laurent and operator-norm completions will likewise remain distinct.

Here are five explicit global inputs. Permutations list darts starting at zero; $E = |H|/2$.

Surface and Belyi map β	σ	α	$(g, \operatorname{rank}_R \Lambda)$
$\mathbb{P}^1$, $4z(1-z)$	(0)(1)	(01)	$(0,2)$
$\mathbb{P}^1$, $(z+1)^2/(4z)$	(01)	(01)	$(0,4)$
$y^2 = x^3 - 1$, x^3	(012)(345)	(03)(14)(25)	$(1,18)$
$\mathbb{P}^1$, $4z^3/(1+z^3)^2$	(012)(345)	(03)(15)(24)	$(0,18)$
$y^2 = x^3 - x$, x^2	(0123)	(02)(13)	$(1,16)$

The first is the nodal interval: $A = k[x,y]/(xy)$ and $\Lambda = k[[x,y]]/(xy)$. The second has $A = k\langle x,y\rangle/(x^2,y^2)$, still infinite dimensional. The two theta dessins have the same two permitted three-cycles but different face cycles, respectively (051324) and (04)(13)(25). The elliptic square has one permitted and one forbidden four-cycle. For example x has pole order two on the cubic $y^2 = x^3 - 1$; x^3 has ramification $(3,3)$ above zero, $(2,2,2)$ above one since $x^3 - 1 = y^2$, and (6) above infinity. The rational theta map has passport $(3,3),(2,2,2),(2,2,2)$. These ramification computations, and the analogous direct zeros/poles in the other rows, verify cellular Belyi inputs rather than merely abstract quivers.

2 Projective resolutions, dual dessins and determinants

Use right modules $P_i = e_i A$, $S_i = P_i/P_i J$. Internal shift $M\langle n\rangle$ places its old generator in degree n. Let $\mathcal{R}_n(i)$ be the *indexing words* of length n from i with every adjacent pair forbidden, including e_i for $n = 0$. Such a word is usually zero in A; it still labels a summand.

Theorem 2.1 (Exact resolutions)**.** *There is a minimal graded projective resolution, with $P_0(i) = e_i A$ and for $n \geq 1$,*

$$P_n(i) = \bigoplus_{p = a_1 \cdots a_n \in \mathcal{R}_n(i)} e_{t(a_n)} A\langle n\rangle, \qquad d_n(x)_p = a_n x \text{ in the summand } a_1 \cdots a_{n-1}.$$

The augmentation $P_0(i) \to S_i$ is the quotient. For the completed order the identical finite-rank projective terms and formal-series maps resolve the residue simple. Ext and Tor against residue simples count forbidden words with the prescribed endpoints. In the graded case Ext has matching internal shift n.

Proof. The maps are right linear and degree zero. Adjacent differentials multiply by a forbidden pair. The augmentation kernel is the direct sum of aA over initial arrows. For any arrow a, multiplication $e_{t(a)}A \to e_{s(a)}A$ kills exactly the permitted paths beginning with its forbidden successor $f(a)$. All other basis paths survive to distinct basis paths. Thus its kernel is $f(a)A$. For a fixed prefix, different last arrows have disjoint surviving supports; there is no cancellation between summands.

This proves exactness inductively. For formal series the same assertion holds coefficient by coefficient and identifies the entire closed kernel, not just its dense polynomial subspace. Terms are idempotent projectives and all entries are arrows. Consequently applying Hom or tensor with a residue simple kills every differential, proving minimality and the Ext/Tor count. In A minimality is graded; it does not require that J be its ordinary Jacobson radical. □

The two arms never terminate in this closed class. Their labelled patterns follow ϕ-cycles. An ungraded positive syzygy tail can have a shorter period by permutation of summands: the nodal interval has labelled period two but unordered tail period one; the elliptic theta has labelled period six and unordered tail period three. A simple need not be isomorphic to a positive syzygy; for the node it is R-torsion, whereas the positive syzygies are lattices.

Let $C_{ij}(q) = \sum_{d\geq 0} \dim_k(e_i A_d e_j) q^d$ and $E_{ij}(q) = \sum_{n\geq 0} (-1)^n \dim_k \operatorname{Ext}^n_A(S_i, S_j) q^n$. Degreewise Euler characteristic of the resolution gives $EC = I$: every coefficient is finite, because a summand in homological degree n starts in internal degree n. Constant terms are I, so the inverse is two-sided. Already signed $E(q)$ must not be evaluated at $-q$ again.

Proposition 2.2 (Dual dessin and cycle determinant)**.** *The complementary-relation algebra is another closed dessin algebra, with $\alpha^{\#} = \alpha$, $\sigma^{\#} = \phi^{-1}$, permitted successor ϕ, and forbidden successor τ. It exchanges graph vertices and faces and preserves genus. Its Cartan matrix satisfies $E(q) = C_{A^{\#}}(-q)$, with the opposite convention if required for Yoneda composition. For independent commuting arrow weights w_a,*

$$\det C_A(\mathbf{w}) = \frac{\prod_{f\in \mathrm{cyc}(\phi)} (1 - (-1)^{|f|} w(f))}{\prod_{c\in \mathrm{cyc}(\sigma)} (1 - w(c))}.$$

Proof. $\alpha(\sigma^{\#})^{-1} = \alpha\phi = \tau$, so the claimed relations and face data follow directly; the connectedness group is unchanged. To prove the determinant, put $S_{v,a} = w_a$ when $s(a) = v$, and $T_{a,v} = 1$ when $t(a) = v$, zero otherwise. On arrow states let $U_{a,b} = w_b$ for permitted adjacency and $V_{a,b} = w_b$ for forbidden adjacency. Then $TS = U + V$ and the path sum is $C = I + S(I - U)^{-1}T$. Sylvester's identity gives $\det C = \det(I + V)/\det(I - U)$. Each transition is a weighted permutation, whose cycle determinant is respectively $1 - (-1)^{\ell} w(c)$ and $1 - w(c)$. Everything holds in formal power series, hence as rational functions. This supplies a direct proof of the Bessenrodt–Holm formula in the closed dessin class [4]. □

The five univariate determinants are, in the order above,

$$\frac{1+q}{1-q}, \quad \frac{1+q}{1-q}, \quad \frac{1-q^6}{(1-q^3)^2}, \quad \frac{(1-q^2)^3}{(1-q^3)^2}, \quad 1.$$

The last cancellation does not imply finite global dimension: its forbidden resolution is infinite. These rational functions have cycle poles/zeros, not arithmetic prime Euler factors.

For clarity, a complex is a sequence with $d^2 = 0$, its homotopy category identifies maps differing by $dh + hd$, and $D(A)$ inverts maps inducing cohomology isomorphisms. A perfect complex is represented by a bounded complex of finite projectives. Our infinite simple resolutions compute derived Hom and tensor but are generally not perfect. The two-term presentation $[P_1 \to P_0]$ is perfect even if its kernel is nonzero. Quadratic linearity is a Koszul statement; it is not by itself an equivalence between all completed derived categories. Likewise a finite-dimensional gentle graded-curve theorem [6] cannot simply be transferred to these closed infinite orders. Here the surface walks explicitly compute the resolution endpoints: τ turns round a graph vertex and ϕ follows a face. For longer string objects all maps can be computed from the occurrence matrices; a universal geodesic-intersection classification for this completed category remains a separate theorem.

2.1 Auxiliary staircase resolutions over the order

The two filtration parameters below are not dessin arrows. Let $R = k[x, y]$ for a field k, and let $I = (g_0, \dots, g_{m-1})$, $m \geq 2$, where $g_i = x^{a_i} y^{b_i}$,

$$a_0 > \cdots > a_{m-1} = 0, \qquad 0 = b_0 < \cdots < b_{m-1}.$$

The positive outer exponents make R/I finite length. Write $R(-a, -b)$ for a free generator born in bidegree (a, b).

Proposition 2.3 (Adjacent syzygies)**.** *A minimal bigraded free resolution of $M = R/I$ is*

$$0 \longrightarrow \bigoplus_{i=0}^{m-2} R(-a_i, -b_{i+1}) \xrightarrow{d_2} \bigoplus_{i=0}^{m-1} R(-a_i, -b_i) \xrightarrow{d_1} R \longrightarrow M \longrightarrow 0,$$

where $d_1(e_i) = g_i$ and $d_2(f_i) = -y^{b_{i+1}-b_i} e_i + x^{a_i - a_{i+1}} e_{i+1}$. Deleting row i from d_2 gives determinant $(-1)^i g_i$. The Hilbert series numerator is $1 - \sum_i s^{a_i} t^{b_i} + \sum_{i=0}^{m-2} s^{a_i} t^{b_{i+1}}$.

Proof. Each column maps to zero. In a relation $\sum h_i g_i = 0$, all terms except the last contain $x^{a_{m-2}}$. Coprimality with $y^{b_{m-1}}$ implies $h_{m-1} = x^{a_{m-2}} q$. Subtract $q d_2(f_{m-2})$ from the coefficient vector; its last component vanishes. Factor $x^{a_{m-2}}$ from the remaining generators. They form a smaller staircase with the same strict inequalities, ending in a pure y-power. Induction expresses the remaining relation in the preceding adjacent columns. The base of one generator has no relations, since R is a domain. Hence $\ker d_1 = \operatorname{im} d_2$. If $d_2(q) = 0$, its first row gives $-y^{b_1} q_0 = 0$; successive rows then force all $q_i = 0$. This proves injectivity. Every differential entry has positive degree, so reduction modulo (x, y) kills the differentials and the resolution is minimal. In the row-deleted matrix the unique nonzero permutation term uses the negative y entries before row i and the positive x entries after it. Their exponents telescope to b_i, a_i, respectively, and the sign is $(-1)^i$. Finally take the degreewise Euler characteristic; the free summands have Hilbert denominator $(1 - s)(1 - t)$. □

For $I = (x^2, xy, y^2)$ the underlying sequence is also kernel checked in Lean 4 over the actual ring MvPolynomial(Fin 2, k). The verification proves the kernel/image equalities, the stated ideal image, surjective quotient augmentation, projective free terms, and vanishing of both induced Hom differentials to the constant-coefficient residue field. It uses regular multiplication on the free modules, distinct from the residue action on k. General staircases, graded minimality and the surface-order coefficient cone remain separately scoped prose arguments.

At grade $\nu \in \mathbb{N}^2$, M_ν has dimension one exactly when no generator degree is componentwise at most ν. Its comparable-grade map has rank one exactly when both endpoint grades satisfy this condition. A CW realization takes one loop and attaches a disk once along it at each generator degree. The active indices form an interval: the x inequality selects a final interval, the y inequality an initial interval. With c active disks, the cellular map $k^c \to k$ has every entry one. Thus $\dim H_1 = \mathbf{1}_{c=0}$ and $\dim H_2 = \max(c - 1, 0)$, the latter counting active adjacent syzygies. This auxiliary CW complex is not the dessin surface.

For the actual nodal order $\Lambda = k[[\xi, \eta]]/(\xi\eta)$ put $u = \xi + \eta$ and $E = \Lambda/u^2\Lambda$. The basis $(1, \xi, u, u\xi)$ of E satisfies $\xi^2 = u\xi$, $u^2 = 0$, $\eta = u - \xi$. Tensoring the resolution above with $0 \to \Lambda \xrightarrow{u^2} \Lambda \to E \to 0$ gives, over $\mathcal{B} = \Lambda[x, y]$, free ranks $(1, m + 1, 2m - 1, m - 1)$. Indeed the scalar resolution stays exact after tensoring over the field; its degree-zero module is free over Λ on the standard monomials. Multiplication by u^2 is injective there and on all free terms, so the long exact homology sequence of its mapping cone gives positive-degree homology zero and degree-zero module $E \otimes_k M$. In contrast, as an R-module this object has ranks $4(1, m, m - 1)$ in its minimal resolution. Neither finite-length E nor these parameter rings replace the infinite parent dessin algebra or its lattice category.

2.2 A monotone quantum comparison with exact finite ranks

Proposition 2.4 (One common channel)**.** *Let T be a positive trace-preserving linear map between finite complex matrix algebras. For density matrices ρ, σ, their trace distance $d(\rho, \sigma) = \frac{1}{2}\|\rho - \sigma\|_1$ satisfies $d(T\rho, T\sigma) \leq d(\rho, \sigma)$. For finitely many labelled states, keep fixed vertex birth grades e_i. At (r, e) include vertices with $e_i \leq e$ and edges with distance at most r, and take the flag complex. Then T gives an identity-on-labels inclusion from the input to the output bifiltration, natural in (r, e).*

Proof. Write the Hermitian difference as $A = A_+ - A_-$ with $A_\pm \geq 0$. Positivity, the trace norm triangle inequality and trace preservation give $\|TA\|_1 \leq \|TA_+\|_1 + \|TA_-\|_1 = \operatorname{tr} A_+ + \operatorname{tr} A_- = \|A\|_1$. Thus every input edge is an output edge; the fixed births preserve vertices, and all cliques are preserved. Increasing either parameter preserves the simplices, and every map is the same label inclusion, so the squares commute. Complete positivity is sufficient but is not needed for this norm argument. □

For a finite inclusion $K \subset L$, work over a field and embed a basis Z_K of source one-cycles into the target edge space. If B_L is the matrix of target two-boundaries, the induced H_1 rank is

$$\rho_1(K, L) = \operatorname{rank}[B_L \; Z_K] - \operatorname{rank} B_L.$$

Its image is $(Z_K + \operatorname{im} B_L)/\operatorname{im} B_L$; source boundaries already map into target boundaries. This proves the formula, including empty complexes. Computing a flag two-skeleton suffices for H_0, H_1, but does not compute the full flag complex's H_2. Integer ceilings of scaled exact distances preserve the inclusion. For numerical distances, frozen integer labels define an exact combinatorial diagram only after their inequality is checked; this is not an interval certificate of the underlying quantum distances. None of these finite ranks identifies a completed arithmetic determinant or proves GRH.

2.3 A correct DG model of the infinite node

Use cohomological degrees and an uncurved differential of degree $+1$. An odd element ε with $d\varepsilon = 1$ contracts the underlying k-complex of every unital left DG module: $h(m) = \varepsilon m$ satisfies $dh + hd = 1$ by the graded Leibniz rule. Thus this example has zero cohomology, contrary to the original ground-field-resolution claim. An isomorphism on H^0 alone is also insufficient for a DG quasi-isomorphism; all degrees must be checked.

Proposition 2.5 (Nodal Koszul model)**.** *Let k be a field, $R = k[\xi, \eta]$ and $f = \xi\eta$ and $K = R \oplus R\varepsilon$, with $|\varepsilon| = -1$, $\varepsilon^2 = 0$, $dR = 0$ and $d\varepsilon = f$. The augmentation $K \to A = R/(\xi\eta)$, $a + b\varepsilon \mapsto [a]$, is a DG-algebra quasi-isomorphism onto the actual infinite nodal dessin algebra. Branch rescaling $\xi \mapsto q\xi$, $\eta \mapsto q^{-1}\eta$, $\varepsilon \mapsto \varepsilon$ for $q \in k^\times$ lifts to a DG automorphism.*

Proof. Multiplication is $(a + b\varepsilon)(c + e\varepsilon) = ac + (ae + bc)\varepsilon$. The differential kills degree zero and sends $b\varepsilon$ to fb. Its square is zero. For two odd inputs the Leibniz right side is $fbc\varepsilon - bfc\varepsilon = 0$; the other homogeneous cases follow by R-linearity. Multiplication by f is injective in the polynomial domain. Hence $H^{-1}(K) = 0$, $H^0(K) = R/(f)$, and all other groups vanish. The augmentation preserves products and kills boundaries. Rescaling fixes f, so its coefficientwise extension commutes with d and has inverse obtained by replacing q with q^{-1}. This is a one-relation Koszul construction [78, 79]. □

Replacing R by $k[[\xi, \eta]]$ gives the completed order: setting each variable separately to zero maps onto the branch pullback, and the kernel consists precisely of mixed monomials, hence equals $(\xi\eta)$. Rescaling is generally semilinear over the coefficient ring, rather than linear over the fixed diagonal

DVR $k[[u]]$, $u = \xi + \eta$. It preserves that DVR as a subring exactly when $q^2 = 1$, and fixes it pointwise exactly when $q = 1$. No Hilbert norm or arithmetic spectrum follows.

The corresponding Lean file checks the actual exterior-line algebra, differential, sign-Leibniz, homotopy and polynomial ambient-resolution identities. Full graded/category/completion objects remain separately scoped. The two theta dessins likewise require endpoint-sensitive Ext: self Ext^1 is zero in both; for the elliptic theta self Ext^3 has dimension two, while for the spherical theta positive even self Ext has dimension two. Outdegree alone does not count self extensions. Minimal A_∞ models use additional graded data [77, §3.3]; m_3 is not automatically a physical curvature.

3 Determinant semi-invariants and Schur constructions

Assume k algebraically closed of characteristic zero for invariant-ring statements. A finite-dimensional representation of the *infinite* A has spaces N_i, arrow matrices $N(a)$ satisfying the monomial relations, and base-change group $G = \prod_i \mathrm{GL}(N_i)$. With column vectors, $N(a) \mapsto g_{t(a)} N(a) g_{s(a)}^{-1}$. Let $P_0 = \bigoplus P_i^{a_i}$, $P_1 = \bigoplus P_i^{b_i}$, and $f : P_1 \to P_0$ be a finite path-matrix map. When $\sum a_i \dim N_i = \sum b_i \dim N_i$, define

$$c_f(N) = \det D_f(N), \qquad D_f(N) = \mathrm{Hom}_A(f, N).$$

It is intrinsically a determinant-line section; chosen bases give this scalar.

Proposition 3.1 (Weights and exact homological scope)**.** $c_f(gN) = \prod_i \det(g_i)^{b_i - a_i} c_f(N)$. *Restricted determinants of this form span all semi-invariants on a closed G-stable relation or nilpotence locus. Set $M = \mathrm{coker}\, f$. On the square locus,*

$$c_f(N) \neq 0 \quad \Longleftrightarrow \quad \mathrm{Hom}(M, N) = 0 \quad \Longleftrightarrow \quad \mathrm{Hom}(M, N) = \mathrm{Ext}^1(M, N) = 0.$$

If f is injective, $\mathrm{coker}\, D_f = \mathrm{Ext}^1(M, N)$ *and higher Ext vanish. Without injectivity neither that cokernel equality nor higher vanishing follows.*

Proof. Naturality gives $D_f(gN) = G_1 D_f(N) G_0^{-1}$, where G_j repeats the relevant g_i. Take determinants. The contragredient coordinate-ring action has the inverse character. Schofield–van den Bergh's spanning theorem allows oriented cycles [36, Theorem2.3]. Exactness of invariants for reductive G in characteristic zero lifts a fixed-weight function from any closed stable locus to the ambient representation space; restrict the spanning determinants. Injectivity of the path map need not survive that restriction. Finally $\ker D_f = \mathrm{Hom}(M, N)$. Writing $K = \mathrm{im}\, f$, the exact sequence with projective P_0 expresses $\mathrm{Ext}^1(M, N)$ as $\mathrm{Hom}(K, N)/\,\mathrm{im}\,\mathrm{Hom}(P_0, N)$; the epimorphism $P_1 \to K$ embeds this quotient in $\mathrm{coker}\, D_f$. A square finite matrix with zero kernel has zero cokernel. If f is injective it is the actual length-one resolution, giving the stronger assertion. □

For the node, $f = \times x$, $M = \Lambda/(x)$, $N = \Lambda/(x, y^2)$ give $\dim \mathrm{coker}\, D_f = 2$ but $\dim \mathrm{Ext}^1(M, N) = 1$. For the elliptic square name arrows so that $a, b : 0 \to 1$, $c, d : 1 \to 0$, permitted cycle *acbd* and forbidden cycle *adbc*. With $f = a$ and $N(a) = 1$, $N(b) = N(c) = N(d) = 0$, its Hom matrix is [1], yet the subsequent forbidden resolution gives $\mathrm{Ext}^{2,3}(M, N) = k$. Thus nonvanishing is not all-degree derived orthogonality.

Proposition 3.2 (What completion does to invariants)**.** *A continuous d-dimensional representation of Λ kills J^d. On the reduced nilpotent representation variety of fixed dimension vector every invariant regular function is constant. Nonzero-weight semi-invariants need not be constant.*

Proof. Continuity with discrete finite-dimensional target kills some J^m; its Loewy filtration has at most d nonzero strict steps. Choose vertexwise complements to this filtration. An arrow sends a layer p into layers $q \geq p+1$. Scaling layer p by z^p conjugates each component by z^{q-p}. At $z = 0$ the representation tends to the zero-arrow representation. Every invariant therefore has the same value at all points; reducedness and the Nullstellensatz give the ring assertion. In the square example the scalar $N(a)$ varies on a nilpotent locus and has weight g_1/g_0, so semi-invariants survive. Formal entries in a projective map are evaluated on this locus by truncating modulo J^d, a legitimate finite polynomial operation. □

On the node, $u = x+y$ is regular and A is free of rank two over $k[u]$. The polynomial presentation $\times(u-\lambda)$ for $\lambda \neq 0$ is injective and its determinant on N is $\det(u_N - \lambda I)$, the characteristic polynomial of u_N up to the sign $(-1)^d$. But $u-\lambda$ is a unit in Λ, with its geometric-series inverse, so those modules do not extend continuously to the completion. An R-free lattice and its u-nilpotent quotients are different categories. On a lattice a separately specified endomorphism F has a meaningful relative characteristic polynomial over R; the scalar base action uI alone does not create Frobenius.

3.1 Young tableaux and a cyclic representation stratum

For a nonempty partition $\lambda \vdash m$, choose a numbering T of its boxes. On $V^{\otimes m}$ let $Y_T = R_T C_T$, the unnormalized row sum followed by the signed column sum. The classical identity $Y_T^2 = h_\lambda Y_T$, where h_λ is the hook product, makes Y_T/h_λ an idempotent in characteristic zero. Its image is $S_\lambda(V)$, one Schur copy, rather than the whole symmetric-group isotypic component [31, 32]. Its character is s_λ, computed by semistandard tableaux (weak rows, strict columns), and

$$\dim S_\lambda(k^d) = \prod_{(i,j)\in\lambda} \frac{d+j-i}{h_{ij}}.$$

Since $f^{\otimes m}$ commutes with the place permutations, it induces $S_\lambda(f)$ and respects composition and zero maps. Applying this at every vertex and arrow therefore gives a genuine representation of the monomial dessin algebra: $S_\lambda(N(b))S_\lambda(N(a)) = S_\lambda(0) = 0$ for a forbidden pair. The empty partition does not have this property: it sends zero arrow maps to the identity of k.

The Littlewood–Richardson rule gives natural intertwiners

$$S_\lambda(V) \otimes S_\mu(V) \cong \bigoplus_\nu S_\nu(V)^{\oplus c^\nu_{\lambda\mu}}.$$

Here $c^\nu_{\lambda\mu}$ counts semistandard tableaux of skew shape ν/λ and content μ whose row-reading word is Yamanouchi. Naturality makes the same decompositions intertwine all the arrows. The tensor operation is vertexwise with synchronous arrow maps, not $\otimes_A$ and not a Hopf-algebra tensor structure. There is no automatic tensor unit preserving these zero relations. These functors are not additive or exact; they cannot simply be applied degreewise to a projective resolution. They also need not preserve the fixed R-action on lattices.

In the elliptic theta set one of the two permitted three-cycle branches to zero. The other has arbitrary maps $X_1 : V_0 \to V_2$, $X_2 : V_2 \to V_1$, $X_3 : V_1 \to V_0$, with all spaces of dimension d. This is a closed representation stratum of our infinite dessin algebra. Put $M = X_3X_2X_1$.

Theorem 3.3 (Cyclic invariants and a local positive kernel)**.** *The invariant ring on this stratum is $k[c_1(M), \ldots, c_d(M)]$, where c_j are characteristic coefficients. Its total-arrow-degree Hilbert series is $\prod_{j=1}^{d}(1-t^{3j})^{-1}$. Common multidegree (m, m, m) has basis $\chi_\lambda(M) = \operatorname{tr} S_\lambda(M)$, $\lambda \vdash m$, $\ell(\lambda) \leq d$;*

other multidegrees have no invariants. Products obey the Littlewood–Richardson rule. Over $\mathbb{C}$*, if* $\|M_i\| \le R_0$ *and* $0 \le r$ *and* $rR_0^2 < 1$*, the kernel*

$$K_r(M_i, M_j) = \det(I - rM_i \otimes \overline{M_j})^{-1} = \sum_\lambda r^{|\lambda|} \chi_\lambda(M_i)\overline{\chi_\lambda(M_j)}$$

is absolutely convergent and positive semidefinite.

Proof. Cycle-trace generation for quivers, including relation quotients, is the Le Bruyn–Procesi theorem [29, 30]. Every cycle here is a power of the full three-cycle, up to cyclic rotation. In characteristic zero Newton identities give the generators c_j. They are algebraically independent: take $X_1 = X_2 = I$ and arbitrary X_3. Scalar changes at the three vertices force equal arrow multidegrees. Restriction to that same section identifies degree (m, m, m) with degree-m symmetric polynomials in eigenvalues, whose Schur basis and product rule give the assertion. For the kernel use the ordinary Cauchy identity, first as a formal identity in eigenvalues. On Schur modules the operator norm of $S_\lambda(M)$ is at most $R_0^{|\lambda|}$, so its trace is bounded by $\dim S_\lambda(\mathbb{C}^d) R_0^{|\lambda|}$. The dimension Cauchy identity bounds the absolute series by $(1 - rR_0^2)^{-d^2}$. Each summand is a rank-one Gram kernel, proving positivity by absolute summation. □

The term for the empty partition is a constant scalar feature, not an allowed module-valued tensor unit. In dimension two,

$$(\operatorname{tr} M)^2 = \operatorname{tr}\operatorname{Sym}^2 M + \det M, \qquad \operatorname{tr}\operatorname{Sym}^2 M = \tfrac{1}{2}((\operatorname{tr} M)^2 + \operatorname{tr} M^2).$$

More generally $\det(I - zM) = \sum_j (-z)^j \operatorname{tr} \bigwedge^j M$ and its reciprocal is $\sum_m z^m \operatorname{tr}\operatorname{Sym}^m M$. These are actual determinant/Schur connections, not identifications with an arithmetic L-function. On continuous finite-length Λ-modules the monodromy is nilpotent and all positive-degree characters vanish. The displayed Cauchy positivity is local representation-theoretic positivity, not the global Weil form.

4 Lattice almost-split sequences and genuine diamond quotients

For the interval dessin let $X = (x)$, $Y = (y)$ in $\Lambda = k[[x, y]]/(xy)$, $R = k[[u]]$, $u = x + y$. A lattice is a finitely generated Λ-module free over R.

Theorem 4.1 (The nodal lattice calculation)**.** *Every lattice is* $\Lambda^r \oplus X^a \oplus Y^b$*. The almost-split sequences in this lattice category are*

$$0 \to Y \hookrightarrow \Lambda \xrightarrow{\times x} X \to 0, \qquad 0 \to X \hookrightarrow \Lambda \xrightarrow{\times y} Y \to 0.$$

In particular $\tau X = Y$*,* $\tau Y = X$*. Multiplication by* u *between two copies of either row, followed by reduction modulo* u*, gives an actual nonzero exact three-by-three diagram.*

Proof. Normalize M *inside its total quotient module*: $\overline{M} = BM$, $B = k[[x]] \times k[[y]]$. The conductor gives $u\overline{M} \subset M \subset \overline{M}$. Both branch projections of $W = M/u\overline{M} \subset k^p \oplus k^q$ are onto. Intersect W with the two factors; its remaining quotient identifies their two residual quotients. Separate bases for these pure factors and the common quotient put W into pure first lines, pure second lines and diagonal pairs. Lift branch bases over the DVRs. These are respectively the summands X, Y, Λ. Their endomorphism rings are local.

Exactness of the first sequence is $\operatorname{ann} x = Y$. It does not split: a map $X \to \Lambda$ lands in $\operatorname{ann} y = X$, and its composite with $\times x$ is a nonunit in $\operatorname{End} X = k[[x]]$. Every map $\Lambda \to X$ factors through

this epimorphism, $\mathrm{Hom}(Y, X) = 0$, and every nonunit endomorphism of X is divisible by x and factors through $X \hookrightarrow \Lambda \to X$. Classification handles all finite sums; a unit X component would be a retraction. This is right almost splitness. The dual argument for nonsections out of Y proves left almost splitness. Endomorphisms fixing either structure map have constant term one, hence are units, proving minimality. Swap x, y for the other sequence. Finally all three terms are R-free. Multiplication by u is injective and reduction preserves exactness, proving the diagram assertion. □

The lattice AR quiver has arrows $Y \to \Lambda \to X$ and $X \to \Lambda \to Y$; the mesh composites are zero. Its projective vertex Λ is not the endpoint of another nonsplit AR sequence. AR meshes do not mean that all rows and columns of any drawn grid are short exact.

For a second, explicit family fix $n \geq 3$ and the rational Belyi map

$$\beta_n(z) = \frac{1}{1 - T_n(z)^2} = \frac{1}{(1 - z^2)U_{n-1}(z)^2},$$

where T_n, U_{n-1} are Chebyshev polynomials. It has passport $(2n), (2^n), (2^{n-1}, 1, 1)$. Its contracted dual graph is a path, identifying its genus-zero dessin. The associated infinite algebra has vertices $1, \ldots, n$, endpoint loops l, r, opposite arrows $a_i : i \to i + 1$, $b_i : i + 1 \to i$, and precisely the relations $l^2, r^2, a_i b_i, b_i a_i$. Its single permitted cycle is $l a_1 \cdots a_{n-1} r b_{n-1} \cdots b_1$; the order rank is $(2n)^2$.

Retain one arrow at each edge in an orientation Ω, and kill its opposite. The quotient is exactly the string algebra $H_n(\Omega)$ of Huang–Lin–Su [46]; the killed-arrow ideal contains J^{n+2}, so it is also an actual quotient of Λ_n. Inflation identifies its module category with the full abelian quotient-annihilator subcategory $\mathcal{C}_\Omega$ of Λ_n-modules. This is generally not extension closed in the whole order category. The imported AR property is intrinsic to $\mathcal{C}_\Omega$, not automatically an ambient lattice or finite-length AR property. The extended edition gives nine explicit string modules, all twelve maps and the four exact meshes from the supplied diamond resource, now situated in this actual dessin quotient.

The other supplied resources also have a precise home. For the interval order the subcategory annihilated by J^2 is $\mathrm{Mod}\, B_0$, $B_0 = \Lambda/J^2$. This is a domestic string quotient, not itself a gentle algebra. Inflation is full and preserves purity, and $\mathrm{Zg}\, B_0$ is a closed subspace of $\mathrm{Zg}\, \Lambda$. Ringel's infinite-word/pure-injective program and its proof by Puninski–Prest apply there [43, 44]; the isolated band xy^{-1} gives Krull–Gabriel and relative Cantor–Bendixson dimension two by [45, Theorem10.19]. This is not an ambient rank calculation for all Λ-modules. The parameter in $x = J_2, y = \lambda J_2$ survives even though all cycle traces vanish; the pp condition $\exists v\ (z = xv\ \wedge\ (y - \lambda x)v = 0)$ detects it. These modules are u-torsion and are not nonzero order lattices. Infinite-string and locally finite AR theorems must retain their distinct finiteness hypotheses; a surviving permitted cycle has infinitely many paths at a vertex.

4.1 Polynomial strings, completed lattices and the Fukaya category

Crawley–Boevey's theorem [38, Theorems1.1–1.4] applies directly to the infinite polynomial dessin algebra: its finitely controlled modules decompose into strings and finite-dimensional bands; its pointwise artinian modules additionally allow primitive injective bands. For our finite quiver, these conditions mean finitely generated and artinian respectively. The words may be infinite, but their modules have algebraic direct-sum bases. This is not a classification of every module or of arbitrary formal completions.

In fact no nonzero completed-order lattice, restricted to A, meets either finiteness condition. The polynomial A is finite over $k[u]$: remove full turns from sufficiently long paths as in Proposition 1.1. If an R-free lattice were finitely generated over A, it would be finite torsion free, hence free, over the

PID $k[u]$. Multiplication by $1-u$ would then fail to be onto, whereas it is a unit over $R = k[[u]]$. Also $M \supsetneq uM \supsetneq u^2M \supsetneq \cdots$ excludes artinianity. For a concrete indecomposable outside the ordinary string/band list take $k[[x]]$ with $y = 0$: its endomorphisms are multiplication by formal series, so its endomorphism ring is local. Its x is injective and nonsurjective, while $1-x$ is invertible; these properties exclude every pure-x ordinary string or band. The extended edition gives the full word argument.

The Fukaya connection requires equally precise categories. Lekili–Polishchuk explicitly allow infinite-dimensional locally gentle algebras, but their equivalence is for *perfect dg modules* of a homologically smooth algebra, with specified grading, line field and stops [66, Definition3.1.1, Lemma3.1.3, Theorem3.2.2]. Every nonempty closed dessin here has both permitted and forbidden cycles, so its polynomial algebra is neither proper nor smooth by their criterion. This does not exclude appropriate singular or completed variants; it prevents a direct use of that particular theorem. Infinite word, graded curve and geodesic are not interchangeable without a surface realization and boundary/grading data.

There is a genuine positive example inside our setting: the polynomial interval quotient $A/(y) = k[x]$ is still infinite dimensional, is smooth, and has no forbidden cycle. Its perfect category has the partially wrapped annulus model with one stop total. In degree zero $\mathrm{Perf}(k[x]) = D^b(\mathrm{mod}_{\mathrm{fg}}\, k[x])$. Inflation of this quotient is fully faithful on ordinary modules, but not on derived categories: $\mathrm{Ext}^2_A(k[x], k[x]) = k$, whereas $\mathrm{Ext}^2_{k[x]}(k[x], k[x]) = 0$. The former follows from the alternating y, x resolution. Thus this quotient bridge does not identify the whole closed dessin's perfect category with the annulus category.

Finally, Leavitt path algebras *can* fail invariant basis number, but need not. The permitted-state Laurent algebras here have finite-dimensional matrix quotients, and therefore invariant basis number. The polynomial and completed dessin algebras have their vertex-simple quotients, giving the same conclusion. The complex Leavitt algebra of G_α below is a dense *-subalgebra of its graph C*-algebra, on that specified graph; it is not the original gentle algebra with ordinary CK1 imposed.

5 Leavitt relations, the constrained Fock algebra and spectrum

Work now over $\mathbb{C}$. Ordinary Leavitt and graph C* constructions [47, 48] cannot simply impose our original forbidden products: if $ab = 0$ and CK1 says $a^*a = p_{t(a)}$, then $0 = a^*ab = b$. Every arrow is a forbidden successor, hence every arrow and then every vertex is zero. CK2 is not needed. The ordinary graph-Toeplitz CK1 setting has the same collapse.

The valid permitted-arrow-state graph instead consists of the disjoint directed σ-cycles. Its Leavitt algebra is $\bigoplus_c M_{|c|}(k[z, z^{-1}])$, and its graph C* algebra is $\bigoplus_c M_{|c|}(C(\mathbb{T}))$. These forget the α-gluing; both theta dessins give the same answer. They are not the completed original order. Nor is the ordinary Toeplitz algebra of a cycle of length $\ell > 1$ equal to $M_\ell(\mathcal{T})$: their K_0 ranks are ℓ and one respectively [50].

Let $\mathcal{H} = \ell^2(\mathcal{P})$ for all permitted finite paths, including vacua e_v. Define $L_a\delta_p = \delta_{ap}$ when the prefix is permitted and composable, and zero otherwise. Let P_v project onto paths starting at v, $Q_a = L_aL_a^*$ and v_v onto the vacuum. Then

$$L_aL_{\phi(a)} = 0, \quad L_a^*L_a = P_{t(a)} - Q_{\phi(a)} = v_{t(a)} + Q_{\tau(a)}, \quad P_v - \sum_{s(a)=v} Q_a = v_v.$$

These follow on basis vectors. The different initial projection avoids CK1 collapse. Write $\mathcal{T}_A = C^*(P_v, L_a)$, a constrained-path Fock C* algebra in the monomial/subproduct setting [51, 52]. The polynomial A acts faithfully: applying a finite path sum to the sum of vacua recovers its coefficients.

This representation need not extend to all formal Λ series: $\sum_{m\geq 0} c^m$ along a cycle would send its vacuum to a nonsquare-summable sum.

Theorem 5.1 (Compact extension and incidence index)**.** *Let $\mathcal{H}_v$ be the terminal-path sector. There is an exact sequence*

$$0 \longrightarrow \bigoplus_{v\in Q_0} \mathcal{K}(\mathcal{H}_v) \longrightarrow \mathcal{T}_A \longrightarrow \bigoplus_{c\in \mathrm{cyc}(\sigma)} M_{|c|}(C(\mathbb{T})) \longrightarrow 0.$$

With $\delta[u] = [1 - W^*W] - [1 - WW^*]$, its boundary map is $\delta(c) = -\sum_{h\in c} e_{[h]}$, minus the unsigned incidence map B_Γ of the cellular graph. Thus, for its E edges,

	$K_0(\mathcal{T}_A)$	$K_1(\mathcal{T}_A)$
Γ *bipartite*	$\mathbb{Z}^{E+1}$	$\mathbb{Z}$
Γ *nonbipartite*	$\mathbb{Z}^E \oplus \mathbb{Z}/2$	0.

Proof. All prefixes preserve the terminal sectors. For paths p, q ending at v, $L_p v_v L_q^* = |\delta_p\rangle\langle\delta_q|$. Therefore the vacuum ideal is exactly the displayed sum of compacts: it contains these matrix units and that sum is itself an ideal. In the quotient, L_a has initial projection $Q_{\tau(a)}$ and range Q_a, a graph CK family for the permitted-state cycles. The Q_a have infinite rank and nonzero quotient images; the length gauge action descends. Gauge-invariant uniqueness [48, Theorem2.1] identifies the quotient as claimed.

For a cycle c based at arrow a, lift its corner unitary by $W = L_c Q_a + (1 - Q_a)$. Then $W^*W = 1$ and $1 - WW^* = Q_a - L_c Q_a L_c^*$ projects onto the first $|c|$ paths starting with a. Their terminal vertices count exactly its incident darts, including twice for a loop. This proves the index formula. The six-term exact sequence gives $K_1 = \ker B_\Gamma$ and $0 \to \operatorname{coker} B_\Gamma \to K_0 \to \mathbb{Z}^{\#\,\mathrm{cyc}\,\sigma} \to 0$. The free last term splits abstractly. Spanning-tree integer elimination gives Smith entries 1 repeated $V - 1$ times and one zero in the bipartite case; an odd cycle instead adds the entry two. This proves the groups. □

The nodal interval has $K_0 = \mathbb{Z}^2$, $K_1 = \mathbb{Z}$; the single-loop sphere has $\mathbb{Z} \oplus \mathbb{Z}/2$, zero. Both theta dessins have $\mathbb{Z}^4$, $\mathbb{Z}$. Coarse K-theory does not recover genus or the arithmetic factors.

5.1 A graph presentation and an actual pullback

The Fock algebra itself has a graph presentation, on a different graph from the original gentle quiver. Let G_α have vertices q_h for $h \in H$ and sinks b_e for $e \in Q_0$, and edges

$$s_h : q_h \to q_{\tau h}, \qquad t_h : q_h \to b_{[\tau h]}.$$

Proposition 5.2. *$C^*(G_\alpha) \cong \mathcal{T}_A$, by $q_h \mapsto Q_{a_h}$, $b_e \mapsto v_e$, $s_h \mapsto L_{a_h} Q_{a_{\tau h}}$ and $t_h \mapsto L_{a_h} v_{[\tau h]}$. The integer graph relation at h is $q_h - q_{\tau h} - b_{[\tau h]}$. Eliminating these relations recovers $K_1 = \ker B_\Gamma$ and $K_0 \cong \operatorname{coker} B_\Gamma \oplus \mathbb{Z}^{\#\,\mathrm{cyc}\,\sigma}$.*

Proof. The two edges at q_h have orthogonal initial pieces and orthogonal ranges summing to Q_{a_h}, by the Fock initial-projection identity. All vertex projections are nonzero. The length gauge acts with degree one on both edge families, so gauge-invariant uniqueness applies. They generate since $L_{a_h} = s_h + t_h$ and $P_e = v_e + \sum_{[h]=e} Q_{a_h}$. In the integer matrix a kernel coefficient is constant around each permitted cycle; its sink equations are exactly $B_\Gamma c = 0$. In the cokernel, eliminate all but one state per cycle; its remaining relation is the sum of its incident sink generators. The remaining cycle generators are free, proving the stated groups. □

For the nodal interval, G_α has two loop vertices, each with an exit to the common vacuum sink. Chirvasitu–Hajac–Tobolski's actual pullback theorem [65, Theorem3.4] gives

$$\{T \in \mathcal{T}_A : \pi(T) \in \mathbb{C}^2\} \cong C^*(E_1), \qquad \mathbb{C}^2 \hookrightarrow C(\mathbb{T})^2 \text{ by constants},$$

where E_1 has the same three vertices and countably many arrows from each loop vertex to the sink, with no loops. The arrow numbered $n \geq 0$ maps to loop^n followed by exit. This algebra is AF, with $K_0 = \mathbb{Z}^3$, $K_1 = 0$. The extended edition verifies every hypothesis. The theorem does not turn arbitrary cycle-vertex identifications into pullbacks: its first graph is acyclic and its path functor must cover all pointed paths. Its sink-extension Theorem4.3 has further injection and path-coverage requirements.

5.2 A separate diagonal-fiber C* completion retaining the pairing

The formal order's residue gluing cannot be copied by taking diagonals of arbitrary full matrices: $\operatorname{diag}(E_{12}E_{21}) \neq \operatorname{diag}(E_{12})\operatorname{diag}(E_{21})$. For $\ell > 1$ there is no nonzero *-homomorphism from $M_\ell(C(\mathbb{T}))$ to a commutative C*-algebra, as restriction to its constant matrix units shows. Restrict the distinguished fiber first. Define

$$D_\ell = \{f \in C(\mathbb{T}, M_\ell(\mathbb{C})) : f(1) \text{ is diagonal}\}, \qquad \mathcal{D}_H = \Big(\prod_c D_{|c|}\Big) \underset{\mathbb{C}^H}{\times} \mathbb{C}^{Q_0},$$

where the first map evaluates diagonal entries at one and the second repeats λ_e at the two darts paired by α.

Theorem 5.3 (A faithful polynomial pullback model)**.** *There is a split exact sequence*

$$0 \to \bigoplus_c C_0(\mathbb{T} \setminus \{1\}, M_{|c|}) \to \mathcal{D}_H \to \mathbb{C}^{Q_0} \to 0,$$

so $K_0(\mathcal{D}_H) = \mathbb{Z}^{|Q_0|}$ *and* $K_1(\mathcal{D}_H) = \mathbb{Z}^{\#\operatorname{cyc}\sigma}$*. The polynomial dessin algebra embeds faithfully and generates this C*-algebra with its adjoints. This embedding does not extend to a unital algebra map from all of* Λ*.*

Proof. The section consists of the constant paired diagonal projections. The kernel is the displayed vanishing fiber, a sum of matrix suspensions, giving the groups by split exactness and suspension. For an explicit embedding set $g(z) = |1 - z|^2/4$, choose one cut arrow per permitted cycle, and send a_h to $g(z)z^{\epsilon_h}E_{h,\tau h}$ in its branch, where $\epsilon_h = 1$ at the cut and zero otherwise. Send e_e to the constant paired diagonal projection. Every forbidden product is zero. A full cycle at h maps to $w_c(z)E_{hh}$, $w_c = g^{|c|}z$. This scalar function is injective on the circle: away from its unique zero at one its argument recovers z. Continuous functional calculus therefore supplies every $f(z)E_{hh}$ with $f(1) = 0$. Multiplying by path images and approximating division away from one supplies every vanishing off-diagonal entry. Together with the constant paired diagonals these generate $\mathcal{D}_H$. Path normal forms in a fixed entry are a nonzero prefactor times a polynomial in w_c; its infinite range proves independence. Evaluation at one first separates the zero-length coefficients. Finally for $z_0 \neq 1$, $1 - w_c(z_0)^{-1}c_h$ is a unit in the formal order, but its proposed image is singular at z_0. A unital map preserves units, proving nonextension. □

This is the C* algebra generated by a faithful polynomial representation and its adjoints, distinct from the completed order and from $\mathcal{T}_A$. The norm closure of the polynomial image alone need not contain those adjoints. Their K groups already differ. The pairing belongs to the labelled

presentation; an unmarked isomorphism class need not recover the ribbon ordering. No arithmetic descent, Frobenius action or completed-L positive determinant is inferred from this additional valid gluing model.

Proposition 5.4 (The degree spectral triple)**.** *Define $N\delta_p = |p|\delta_p$ on $\{\xi : \sum_p |p|^2|\xi_p|^2 < \infty\}$. With the polynomial creation/annihilation *-algebra this is an ungraded spectral triple, p-summable exactly for $p > 1$. With N_+^{-s} zero on the kernel,*

$$\operatorname{Tr} e^{-tN} = E + \frac{2E}{e^t - 1}, \quad \operatorname{Tr} N_+^{-s} = 2E\zeta(s), \quad \operatorname{Tr}\big(\chi(N)N_+^{-s}\big) = 2EL(s,\chi)$$

for $t > 0$ and $\operatorname{Re} s > 1$ in the last two formulas, where χ is an externally chosen Dirichlet character.

Proof. There are E vacua and $2E$ paths of every positive length: each starting arrow has a unique continuation. The real diagonal operator is self-adjoint on the stated maximal domain; finite supports are a core and its resolvent is compact. Prefixes preserve the domain and $[N, L_a] = L_a$, $[N, L_a^*] = -L_a^*$, so polynomial commutators are bounded. The trace formulas and summability reduce to the corresponding scalar series. The character multiplier is bounded; membership in $\mathcal{T}_A$ is not asserted. □

Length parity anticommutes with the creators but commutes with N, so it is not an even-spectral-triple grading. The ordinary odd Fredholm class of this positive N is trivial: $N(1+N^2)^{-1/2} - I$ is compact. The spectrum depends only on E; zeros of the scalar zeta trace are not eigenvalues of N.

6 Arithmetic attachment, descent and the remaining comparison

The compact ideal gives a precise home for an arithmetic determinant, with no claim that the arithmetic is derived from the dessin. Fix one terminal sector. Its two paths of each length $n \geq 1$ give orthonormal vectors $\xi_n = (\delta_{p_{n,1}} + \delta_{p_{n,2}})/\sqrt{2}$ and an isometry $U : \ell^2(\mathbb{N}_{\geq 1}) \to \mathcal{H}_v$.

For a nonprincipal character χ put $A_n = \sum_{j \leq n} \chi(j)$, $B_0 = \sup |A_n| \leq q_\chi$, and, on $2^k \leq n < 2^{k+1}$, set

$$w_n = \sqrt{n}(k+1), \quad u_n = A_n/w_n, \quad v_n(s) = w_n(n^{-s} - (n+1)^{-s}), \quad \varphi_s(z) = \sum_n v_n(s) z_n - z_1.$$

Here $u_1 = 1$. For $\operatorname{Re} s > 0$, $u, v(s) \in \ell^2$ and the pairing is complex linear, with no conjugation of s. Define

$$\widetilde{K}_s = U(u \otimes \varphi_s)U^*, \qquad \det_F(I + \widetilde{K}_s) = L(s,\chi).$$

Indeed dyadic blocks give $\|u\|^2 \leq 2B_0^2$, and on a compact set $\operatorname{Re} s \geq \sigma_0 > 0$, $|s| \leq M$, the squared v tail is bounded by $M^2 \sum_{k \geq J} (k+1)^2 2^{-2\sigma_0 k}$. Thus the rank-one family is trace-norm holomorphic. Its determinant is $1 + \varphi_s(u)$, equal by Abel summation to L. Replacing v by $P_N v$ and adding $A_N(N+1)^{-s}e_1^*$ to the functional gives exactly every raw partial sum S_N, with compact-uniform trace-norm convergence; $\|UXU^*\|_1 = \|X\|_1$. For primitive χ, completion replaces the functional by $g(s)\sum v_n(s)z_n - z_1$, with the standard nonzero gamma factor g, rather than multiplying the whole old functional. The extended edition supplies the quantitative estimates.

This inserts a scalar encoding into one actual dessin compact ideal; it is not known to be A-linear, a derived chain map or Frobenius compatible. A common fixed positive metric cannot make this same fixed-range family symmetric for all critical heights: symmetry forces its functionals to be proportional, whereas the coefficient ratio at heights $0, t \neq 0$ is asymptotic to $(\sigma + it)n^{-it}/\sigma$, with two opposite subsequential limits. Different positive realizations remain possible.

Proposition 6.1 (Complex dessin data do not determine arithmetic descent)**.** *The rational Belyi pairs $E_d : y^2 = x^3 - d$, $\beta_d = x^3/d$, for $d = 1, 2$, are isomorphic over $\mathbb{C}$ but have distinct Frobenius polynomials at the good prime seven: $1 - 4T + 7T^2$ and $1 - T + 7T^2$.*

Proof. Choose $a^3 = d$, $b^2 = d$. The projective isomorphism $(x, y) \mapsto (ax, by)$ from E_1 to E_d preserves the Belyi map. The degree-six clean passports are the elliptic theta passport already computed. Directly counting y for $x = 0, \dots, 6$ gives respectively $(0, 1, 1, 0, 1, 0, 0)$ and $(0, 0, 0, 2, 0, 2, 2)$. Including infinity yields four and seven points, hence $a_7 = 8 - \#E_d(\mathbb{F}_7) = 4, 1$. Both discriminants $-432d^2$ are nonzero modulo seven. □

Any invariant of the complex dessin or its undecorated order/Fock algebra alone is the same for these two inputs, so cannot recover both factors without descent data. This does not obstruct a theory including a specified rational model and compatible Galois action. Elliptic H^1 has weight one and unshifted center $s = 1$, unlike Dirichlet center $1/2$; the example is not a counterexample to GRH.

6.1 A constructive arithmetic benchmark on the theta dessin

The preceding twists can be distinguished constructively once their finite-field descent data are retained. This closes the elementary local-factor computation, not the global positive-determinant problem. Let $\kappa = \mathbb{F}_q$, $\operatorname{char} \kappa \neq 2, 3$ and $d \in \kappa^*$. Put $E_d : y^2 = x^3 - d$ and $a_q(d) = q + 1 - \#E_d(\kappa)$. For $q \equiv 1 \pmod 3$, fix a nontrivial cubic character ρ and the quadratic character η, both extended by zero at zero. Define

$$J = \sum_{u \in \kappa} \rho(u)\eta(1 - u), \qquad j = -\eta(-1)J, \qquad \lambda_d = \eta(d)\rho(d), \quad \pi_d = \lambda_d j.$$

Theorem 6.2 (Uniform local factor)**.** *For every such prime power and nonzero d,*

$$a_q(d) = \pi_d + \overline{\pi_d}, \quad |\pi_d|^2 = q, \quad P_d(T) := 1 - a_q(d)T + qT^2 = (1 - \pi_d T)(1 - \overline{\pi_d} T).$$

If $q \equiv 2 \pmod 3$, instead $a_q(d) = 0$. The split-case values lie in $\mathbb{Z}[\omega]$, $\omega^2 + \omega + 1 = 0$.

Proof. Cube fibers have size $1 + \rho(t) + \overline{\rho(t)}$, including at zero; the first term is the literal constant one, not the zero-extended trivial character. Counting square fibers gives $a_q(d) = -\sum_x \eta(x^3 - d)$. Grouping by $t = x^3$ removes the constant term since $\sum_t \eta(t - d) = 0$. Substitution $t = du$ gives $\sum_t \rho(t)\eta(t - d) = \rho(d)\eta(-d)J$; the other term is its conjugate. This proves the signed trace formula.

For a nontrivial additive character ψ, write $G(\chi) = \sum_x \chi(x)\psi(x)$. Character orthogonality gives $|G(\chi)|^2 = q$ for nontrivial χ: substitute $x = uy$ in its squared norm; the inner sum over $y \neq 0$ is $q - 1$ at $u = 1$ and -1 otherwise. Grouping $G(\rho)G(\eta)$ by $x + y$ gives $JG(\rho\eta)$, the zero-sum fiber vanishing since $\rho\eta$ is nontrivial. The three Gauss norms yield $|J|^2 = q$. Neither purity nor RH was assumed. The remaining signs and character values have modulus one. For $q \equiv 2 \pmod 3$ cubing is bijective, so the quadratic-character sum is zero. Finally all split-case summands are signs times cube roots of unity. These are the classical Gauss–Jacobi methods [68, pp.499–502], with our normalization proved explicitly. □

At $q = 7$, choose $\rho(3) = \omega$. Then $j = 3 + 2\omega$, $\lambda_2 = \omega^2$, and $\pi_2 = -1 - 3\omega$, giving the two factors already counted. At $q = 13$, $\rho(2) = \omega$ gives $J = 1 + 4\omega$ but $j = -1 - 4\omega$; this detects the essential sign hidden by the first example.

6.2 A genuine cellular carrier, and what still fails for the order

Assume $q \equiv 1 \pmod 3$ in this cellular construction. The elliptic theta dessin has cellular chains $C_2 = \mathbb{Q}$, $C_1 = \mathbb{Q}^3$, $C_0 = \mathbb{Q}^2$. Orient the three edges from v_+ to v_-; then $\partial_1(e_i) = v_- - v_+$ and $\partial_2 = 0$. Thus H_1 is the sum-zero plane, with basis $e_0 - e_1, e_1 - e_2$. Rotation σ acts there by

$$C = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad G = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

G is the actual Euclidean edge Gram matrix in this basis. The deck involution α acts as $-I$. Consequently the six-element deck group $\langle \sigma, \alpha \rangle$ acts on this homology and permutes the original quiver and its entire projective resolution.

Write $\pi_d = A_d + B_d\omega$. Its multiplication action on this *actual cellular* plane is

$$M_d = A_d I + B_d C = \begin{pmatrix} A_d & -B_d \\ B_d & A_d - B_d \end{pmatrix}, \qquad \det(I - TM_d) = P_d(T), \qquad M_d^t G M_d = qG.$$

This constructs the local matrix from finite-field character sums and the dessin's deck action, rather than fitting its trace. Positivity follows from $2x^2 - 2xy + 2y^2 = (x-y)^2 + x^2 + y^2$. The normalized matrix is G-unitary, not asserted self-adjoint. The extended edition supplies a rational cellular chain lift, its exact determinant cancellation, and Kummer descent compatibility on homology. Its odd-over-even determinant ratio is the classical finite-field curve zeta function

$$Z(E_d/\kappa, T) = \frac{P_d(T)}{(1-T)(1-qT)}.$$

Identification with the geometric ℓ-adic Frobenius polynomial uses the standard elliptic trace and determinant theorem; a canonical integral comparison or an order-derived Frobenius functor has not been constructed here.

There is also a precise obstruction to extending this same real two-dimensional carrier to $q \equiv 2 \pmod 3$. Frobenius then conjugates the cubic deck action. The relation $MC = C^2M$ forces $M = \left(\begin{smallmatrix} a & b \\ a+b & -a \end{smallmatrix}\right)$, whose determinant is $-a^2 - ab - b^2 \le 0$, contradicting the required positive determinant q. This excludes that real carrier with its fixed deck action, not complex or enlarged Hilbert–Pólya constructions. The extended edition gives a compatible complex local replacement.

For the following insertion test use the characteristic-zero theta order $\Lambda_{\mathbb{Q}}$ over $R_{\mathbb{Q}} = \mathbb{Q}[[u]]$, with residue simple $S_e = \mathbb{Q}$. The limitation is concrete. The deck action has finite order and cannot alone supply the norm-$\sqrt{q}$ Jacobi coefficient. Inserting M_d into $S_e \otimes \mathbb{Q}^2$ gives an actual residue-module endomorphism with determinant P_d, but that simple is not perfect over the order and choosing it breaks full deck symmetry. Inserting the same matrix $\Lambda_{\mathbb{Q}}$-linearly into the regular projective instead gives the $R_{\mathbb{Q}}$-determinant P_d^{18}, or P_d^6 on $e\Lambda_{\mathbb{Q}} \otimes \mathbb{Q}^2$. Taking a root is not an established determinant-line comparison; a deck-semilinear lift is a different map. These surviving multiplicity/equivariance obligations prevent promoting the local computation to the required arithmetic order functor.

For the global comparison fix a nonzero rational d and use its good reductions. The finite-field curve zeta has center $1/2$ after $T = q^{-s}$. The *global elliptic* L-function over $\mathbb{Q}$ has unshifted center one, whereas Dirichlet L-functions have center $1/2$. Local weight-one purity does not determine either global zero set. The elementary local bound proves absolute convergence of the elliptic Euler product on $\operatorname{Re} s > 3/2$; the new local identities give no determinant-preserving continuation with the desired global positive spectrum.

6.3 Arbitrary projective actions: an obstruction and a genuine determinant repair

The preceding tensor test has a stronger replacement. Keep the characteristic-zero completed elliptic theta order Λ over $R = \mathbb{Q}[[u]]$, and write $K = \mathbb{Q}((u))$. All determinants in the next theorem are *ordinary underlying-R determinants*; they are not reduced norms.

Theorem 6.3 (Projective determinant multiplicity)**.** *Let P be any finitely generated projective right Λ-module and $f \in \mathrm{End}_\Lambda(P)$. Then $P \simeq \bigoplus_{i=0}^{2} (e_i\Lambda)^{n_i}$. Let $A_i \in M_{n_i}(\mathbb{Q})$ be the induced maps on its three top multiplicity spaces. Then*

$$\det_R(1 - Tf)\Big|_{u=0} = \left(\prod_{i=0}^{2} \det(1 - TA_i)\right)^6.$$

For any bounded complex of these projectives and any Λ-linear chain endomorphism, its alternating determinant ratio specializes to a sixth power in $\mathbb{Q}(T)$. Generically, the ordinary ratio is a cube in $K(T)$.

Proof. The arrow ideal is the Jacobson radical: its elements admit geometric-series inverses after subtracting from one, and the quotient is $\mathbb{Q}^3$. A projective's top lifts to a map from the displayed sum; Nakayama gives surjectivity, projectivity splits it, and another application of Nakayama kills the kernel.

Put $B = \Lambda/u\Lambda$. Each e_iB has path layers of dimensions $1, 2, 2, 1$: the identity, two length-one paths, two length-two paths, and one length-three path. The two full cycles based at i add to ue_i, so become negatives; all length-four paths vanish. A positive-length entry of a projective homomorphism strictly raises path length. Consequently the associated graded map has six copies of $A = \mathrm{diag}(A_0, A_1, A_2)$, giving the formula by triangular block determinants. Base change of determinants justifies specialization. Multiplicativity along filtrations is ordinary linear algebra, consistent with [69, Tags0FJ9,0GSX]; no purity assumption enters.

Each term's determinant has constant term one in T, so every denominator has nonzero specialization. Taking the finite alternating product therefore preserves the sixth power, even after cancellation. Finally $\Lambda_K = M_3(K) \times M_3(K)$; a projective's branch modules are sums of three-dimensional simples. A commuting endomorphism acts on each multiplicity space, repeating every factor three times. This proves the generic assertion. □

Thus no such ordinary perfect-complex ratio can equal the constant-in-u quadratic $P_d(T)$, or the full local curve zeta $P_d(T)/((1-T)(1-qT))$. For P_d, the valuation at infinity is -2, not a multiple of six. For the zeta ratio, $P_d(1) = q + 1 - a_q > 0$ by the local bound $|a_q| \le 2\sqrt{q} < q + 1$, so its pole at $T = 1$ is simple. This also covers square prime powers where the numerator may have a double root. It is an obstruction for the specified ordinary determinant and linear category, not all spectral or determinant theories.

There is nevertheless a *genuine integral functor* removing this multiplicity. Choose a permitted branch and its cycle cut. Its projection $\Lambda \to B_+ \subset M_3(R)$ makes the column lattice $W_+ = R^3$ a left Λ-module. Define

$$\Phi_+ : \mathrm{Perf}(\Lambda) \longrightarrow \mathrm{Perf}(R), \qquad P^\bullet \longmapsto P^\bullet \otimes_\Lambda W_+.$$

Bounded projectives compute this derived tensor functor; flatness of W_+ on all modules is unnecessary. Since $e_iW_+ \simeq R$, its determinant on $e_i\Lambda \otimes_{\mathbb{Q}} V$ with external action $1 \otimes M_d$ is exactly P_d, whereas on the regular module it is P_d^3. This is a defined change of determinant functor, not an arbitrary

polynomial root. It does not construct the arithmetic action independently: the already computed M_d is still the input.

The price of this repair is precise. Branch swap α exchanges W_+ and W_-; pairing them yields P_d^2 on that basic-projective input. The chosen branch determinant cannot intertwine α on all endomorphisms: the corner unit $e_i + c_i^+$ acting on $e_i\Lambda$, with c_i^+ a plus full cycle, acts on its determinant by $1 + u$, whereas its α-image acts by one. Also no rank-one-per-vertex column lattice supported on one branch admits an R-linear lift of the rotation σ fixing u. Its three nonzero arrow scalars have nonnegative u-valuations summing to one; an invertible cyclic lift would make all three equal. This contradiction concerns that lattice, not every determinant-level rotation comparison. Ramification $u = v^3$ *and a new lattice* with all three arrow scalars v admits such a rotation, changing the stipulated base/lattice data. A descent-compatible arithmetic determinant retaining the required symmetry and a global positive realization remains open.

6.4 Tame descent: a surviving order and a sharper obstruction

We now test the proposed ramified repair rather than assume that its rotation descends. Extend coefficients explicitly to $k = \mathbb{Q}(\omega)$, $\omega^2 + \omega + 1 = 0$. Put $R = k[[u]]$, $S = k[[v]]$, $u = v^3$, and let $g(v) = \omega v$, fixing k. On the balanced plus column $W = S^3$, let $C\epsilon_i = \epsilon_{i+1}$ and

$$\Theta = Dg, \qquad D = \operatorname{diag}(1, \omega^2, \omega).$$

The plus arrows act by $vE_{i,i-1}$, and minus arrows by zero. Direct calculation gives $\Theta^3 = 1$, compatibility of Θ with this Λ_R-action, and

$$\Theta C = \omega^2 C\Theta, \qquad L = W^\Theta = R\epsilon_0 \oplus Rv\epsilon_1 \oplus Rv^2\epsilon_2.$$

Thus invariants restore precisely the original cut lattice. The extension $R \to S$ is finite *free* of rank three and faithfully flat; the group quotient is not a torsor at $v = 0$. The comparison $S \otimes_R L \to W$ has matrix $\operatorname{diag}(1, v, v^2)$, determinant u and cokernel $S/(v) \oplus S/(v^2)$. Hence this semilinear group action alone is not effective descent of the column to R. This is the explicit local root-stack situation, consistent with the trivial-stabilizer criterion for vector-bundle descent [70, Theorem10.3]; no stack theorem is needed for the displayed calculation.

Two positive constructions survive. The determinant line $\bigwedge_S^3 W$ has trivial D and C actions and does descend, but the natural map

$$\det_R(L) \longrightarrow (\det_S W)^\Theta$$

has image uR inside R, not an isomorphism. Determinant and coarse invariants cannot be interchanged while dropping this nonunit comparison. Also the *carrier* endomorphism algebra $\operatorname{End}_S(W)^\Theta$, conjugated to the invariant basis, is exactly the hereditary order with entries R on and below the diagonal and uR above. Its rotation descends as $\operatorname{Ad}(A)$, where

$$A = \begin{pmatrix} 0 & 0 & u \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A^3 = uI.$$

The algebra automorphism has order three although A is not an integral unit. This is not $\operatorname{End}_\Lambda(W)$ or a descended arithmetic cohomology group.

The natural arithmetic attempt fails: $aI + bC$ does not commute with Θ when $b \neq 0$, as in both prime-seven Jacobi examples. The original cellular sum-zero plane is not Θ-stable either. Replacing C by vC does give the descended operator A on L, but

$$\det(1 - T(aI + bA)) = (1 - aT)^3 - ub^3T^3,$$

whose special fiber is $(1 - aT)^3$, not the required local quadratic. These carrier maps are not automatically Λ-linear arithmetic endomorphisms.

There is a stronger category-level conclusion.

Theorem 6.4 (Full-deck coefficient kernels)**.** *Let W_0 be a finite R-free left Λ_R-module on which the central u acts by the given scalar. Suppose it has invertible R-linear intertwiners lifting the order automorphisms σ and α. Then, for some $m \geq 0$, every e_iW_0 has rank $6m$. For any finite projective P and Λ_R-linear $f : P \to P$, let A_i be its top blocks. The genuine coefficient functor $\Phi_{W_0} = - \otimes^{\mathbf{L}}_{\Lambda} W_0$ satisfies*

$$\det_R(1 - T\Phi_{W_0}(f))\Big|_{u=0} = \left(\prod_i \det_k(1 - TA_i)\right)^{6m}.$$

Bounded perfect ratios retain this exponent. Thus this whole class of full-deck, fixed-center finite free coefficient kernels cannot supply either the single local quadratic or the full local curve zeta.

Proof. In the generic fiber intersect W_0 with its two central branch summands to obtain finite free integral lattices $W_\pm$; an integral direct-sum decomposition is not assumed. Rotation preserves them and branch swap exchanges them. On a branch let the common vertex rank be r. The three arrow matrices have cyclic product uI_r. Their determinant valuations are nonnegative, sum to r, and rotation intertwiners make them equal. Therefore $3 \mid r$. Branch swap makes the two ranks equal, so the full vertex rank is $6m$.

For the determinant formula, filter W_0/uW_0 by the arrow radical, whose fourth power vanishes. A positive-length entry of f raises the filtration; a top block A_i repeats across its i-th graded pieces a total of $\dim_k e_i(W_0/uW_0) = 6m$ times. Triangular determinants give the assertion. Finite alternating products preserve it. Quadratic degree and the uncancelled simple pole of the full local zeta exclude the targets as before. □

This theorem is sharp: $W_0 = \Lambda_R$ has vertex rank six. It requires neither left projectivity nor left flatness of a general kernel. It excludes this coefficient category, not arbitrary semilinear correspondences, twisted determinant lines or infinite spectral constructions. The extended edition identifies the full character-orbit repair with the hereditary normalization and explains why Schur commutator cancellation is insufficient. All determinant multiplicities here are relative to the stated k-valued base; restricting a rational quadratic factor to the former $\mathbb{Q}$-base squares it. No arithmetic descent or global positive determinant has been recovered by this ramification.

The recent NCG literature supplies concrete comparison targets, with strong hypotheses. Connes–Consani construct finite abelian covers of the adelic Riemann sector with prime-orbit length $\log p$ and arithmetic Frobenius monodromy [53, Theorem3.12]. The prolate/local-factor and moment constructions [54, 55] give particular finite-prime analytic models, not a uniform positive comparison for all completed L-functions. The zeta-spectral-triple determinant theorem [56, Theorem5.10] assumes a specified restricted Weil form, simple even bottom eigenfunction and boundary normalization. Its infinite comparison is still an additional step. The distributional real-zero theorem [57, Theorem6.1] concerns a particular interval quadratic form, essential self-adjointness on its trigonometric core and an even simple ground state; a Fock vacuum alone verifies none of those arithmetic identifications.

Deninger's periodic-packet construction and real-cohomology obstruction [60, 62] further specify, rather than remove, these issues. Full theorem-level reading coverage and exact versions are recorded alongside this draft.

A complete approach must construct arithmetic descent and prime actions on actual order/derived or analytic objects, prove determinant-line and local-factor compatibility, control regularization and conductor/gamma normalization, and establish a positive realization for that *same* function. Finite-field cohomological trace formulas and Deligne purity supply the classical model [20, 21]; alternating duality alone is not positivity or purity. In a number-field construction these are obligations, not imported conclusions.

For a sharply testable final analytic target, put $F_\chi(z) = \Lambda(1/2 + iz, \chi)$. Suppose entire functions $h_m(z)\det(zI - H_m)$, with H_m finite self-adjoint and h_m entire nowhere zero, converge locally uniformly on $\mathbb{C}$ to this nonzero completed function. Hurwitz on every disk disjoint from $\mathbb{R}$ then proves all its zeros real, hence the Dirichlet critical-line assertion. Establishing this convergence with independently constructed H_m is exactly the missing load-bearing claim; a rank-one encoding or a reformulation does not establish it. Hecke, Dedekind and Artin extensions must state their analytic scope anew.

6.5 Whole components, derived endpoints and determinant relations

The full node lattice category has precisely the three indecomposables X, Λ, Y and arrows $X \to \Lambda \to Y$, $Y \to \Lambda \to X$. Its stable category has only X, Y, no irreducible arrows, and suspension exchanging them; its AR triangles have zero middle term. For the polynomial node at a smooth support $x = \lambda + t$, $y = 0$, $\lambda \neq 0$, the modules $T_r = k[t]/t^r$ form an ambient homogeneous tube:

$$0 \longrightarrow T_r \xrightarrow{(\pi,t)} T_{r-1} \oplus T_{r+1} \xrightarrow{(t,-\pi)} T_r \longrightarrow 0, \quad T_0 = 0, \quad \tau T_r = T_r.$$

The shift changes the derived placement, not the quasi-length. These nonzero-central-support objects do not extend to continuous finite-length modules of the arrow-adic completion. At its singular point even the candidate pure-branch tube fails ambient almost-splitness: $T_2^y \to k$ cannot factor through $T_2^x \to k$, since every intervening map lands in the x-socle, killed by the last map.

The nine HLS strings above sit in an entire intrinsic quotient component $T_{p,q}$, $(p, q) \in \mathbb{Z}^2$, with arrows increasing one coordinate and $\tau T_{p,q} = T_{p-1,q-1}$ [46]. Its central object $E = M(ab)$ has infinite projective dimension: restriction to the first corner gives the dual-number simple with its infinite loop resolution, while restricted projectives are free or zero. Its module AR sequence therefore yields a distinguished derived triangle but not a derived AR triangle ending at E. Ringel's four-quarter wind-wheel tile [43], with infinite-word diagonals and mixed center, is a different category's quilt; it is not silently identified with that HLS component or the Belyi surface.

There is no general equation "AR mesh equals polynomial semi-invariant syzygy." The intrinsic $k(1 \to 2)$ quotient of the infinite Chebyshev algebra has AR row $0 \to S_2 \to E \to S_1 \to 0$ and a free presentation determinant z on dimension $(1, 1)$, so its semi-invariant ring $k[z]$ has no generator relation. A genuine framed elliptic-square stratum has $Z = [x_1, x_2, y_1, y_2] \in \mathrm{Mat}_{2\times 4}$ with six minors and relation

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0.$$

These minors have the same determinant character for the target GL_2 with framed source; they are not all semi-invariants for the full unframed vertex group. In contrast, the nodal lattice AR row gives a genuine relative determinant identity over R: multiplication by $a + bx$ on the basis $(y, 1)$ has matrix $\left(\begin{smallmatrix} a & -b \\ 0 & a+ub \end{smallmatrix}\right)$ and characteristic factor $(1 - Ta)(1 - T(a + ub))$. The endpoints have infinite order-projective dimension but are R-perfect.

Finally, the three-vertex theta algebra has no unital surjection onto a six-vertex radical-square-zero circular algebra: its semisimple image has at most three primitive idempotent images. Duplicating face vertex spaces instead gives an exact faithful unfolding which is not full; it does not inherit quotient or AR-transfer claims. Zwara Hom-control means that the Hom-dimension defect is bilinear on K_0, not merely a covering direct-sum formula [72, 73]. The book supplies explicit fixed-dimension counterexamples, corrected circular-complex incidence resolutions, complete drawn component windows and all category hypotheses. These additions do not supply the arithmetic determinant comparison.

7 A genuine theta covering and its categorical limits

Use column matrices, hence covariant quiver representations (left modules); opposite categories translate the right-module convention in [74]. Let A be the infinite elliptic theta dessin algebra: vertices $i \in \mathbb{Z}/3$, arrows $a_i, b_i : i \to i+1$, and all mixed length-two paths zero. For the arrow ideal J and $L \geq 2$ put $B_L = A/J^L$. This is a finite string quotient of our actual infinite gentle algebra; for $L > 2$ its length-L relations prevent treating it as quadratic gentle.

Proposition 7.1 (A locally bounded integer cover)**.** *The category $\widetilde{B}_L$ with vertices $r \in \mathbb{Z}$, two arrows $a_r, b_r : r \to r+1$, and the same mixed and length-L zero relations, has orbit B_L under $r \mapsto r+3$. For finite-support representations, push-down F is exact and*

$$\operatorname{Hom}_{B_L}(FM, FN) \simeq \bigoplus_{g \in \mathbb{Z}} \operatorname{Hom}_{\widetilde{B}_L}(M, T_g N), \qquad (T_g N)_r = N_{r-3g}.$$

Only finitely many summands are nonzero. In general F is neither full, dense, nor Hom-controlled in Zwara's sense.

Proof. A nonzero path is constant-branch with length less than L. Thus the incoming and outgoing total Hom dimensions at a cover vertex are $2L-1$, and $\dim B_L = 3(2L-1)$. Paths and relations lift uniquely; the deck action is free and has the stated orbit. Push-down sums spaces of equal residue. A morphism decomposes uniquely into blocks with target lift index minus source index equal to $-3g$. The arrow commutation equations preserve that difference, proving the formula; finite supports bound the possible differences. Direct sums of vector spaces preserve exact sequences. The following explicit witnesses prove the three failures. □

For $L = 5$, let P be the cover interval on $0, 1, 2, 3, 4$ with all its a-maps one and b-maps zero, and let S be the sum of the five simples at the same vertices. They have the same K_0 class, but

	$\dim \operatorname{End}_{\widetilde{B}_5}$	$\dim \operatorname{End}_{B_5}(F(-))$	defect
S	5	9	4
P	1	2	1

Indeed the pushed-down simple multiplicities are $(2, 2, 1)$, whereas $\operatorname{End}_{B_5}(FP) = kI \oplus kX^3$ for its single nilpotent five-step chain X. The extra endomorphism X^3 proves non-fullness; the unequal defects at the same class exclude a bilinear K_0 defect. To use Zwara's finite-algebra definition literally, restrict the source to the finite convex interval $[0, 4]$, then extend by zero before push-down. This exact functor has the same endomorphism spaces and K_0 comparison.

Proposition 7.2 (An indecomposable module outside this cover image)**.** *Over a characteristic-zero field, B_3 has a nine-dimensional indecomposable representation which is not the push-down of any finite-support $\widetilde{B}_3$ representation.*

Proof. Label its basis by $0, \dots, 8$ with vertex residues $(0, 1, 0, 1, 2, 1, 2, 0, 2)$, and use matrix units E_{ij} acting on columns. Set

$$X = E_{10} + E_{32} + E_{43} + E_{65} + E_{76} + E_{08},$$
$$Y = E_{12} + E_{45} + E_{78}.$$

Their restrictions between adjacent vertex spaces give the a_i, b_i maps. They satisfy $XY = YX = X^3 = Y^3 = 0$, hence every length-three path is zero. The endomorphism ring preserving vertex spaces is $kI \oplus \mathfrak{r}$, where

$$\mathfrak{r} = \langle E_{02} + E_{13}, E_{15}, E_{35} + E_{46}, E_{48}, E_{68} + E_{70}, E_{72} \rangle, \qquad \mathfrak{r}^3 = 0.$$

Solving the arrow commutation equations gives exactly these seven basis elements. The ring is local: an element $cI + n$ is invertible exactly when $c \neq 0$, by the finite geometric series. It has no nontrivial idempotents, so the representation is indecomposable.

A finite-support lift would define an Euler operator D acting by r on the summand at lift vertex r. Then $[D, X] = X$ and $[D, Y] = Y$. Every displayed nonzero arrow is the unique nonzero entry in its row and column in its own matrix. Its commutator entry is therefore $D_{ii} - D_{jj}$. Add the six X equations and subtract the three Y equations. Each diagonal term cancels, leaving $0 = 6 - 3 = 3$, a contradiction. This argument allows arbitrary D, not just a diagonal matrix in the displayed basis. □

Corollary 7.3 (Higher-genus and derived non-density). For the clean dessin with rotations $\sigma = (0 \cdots m-1)(m \cdots 2m-1)$ and $\alpha = \prod_{i=0}^{m-1}(i\, i+m)$, odd $m \geq 3$, and genus $(m-1)/2$, the analogous A_m/J^3 has a $3m$-dimensional representation outside its integer-cover image in characteristic zero. Its bounded-derived push-down is not dense either.

Proof. Use basis $v_0, \dots, v_{3m-1}$, assign vertices $i, i+1, i$ to $v_{3i}, v_{3i+1}, v_{3i+2}$ modulo m, and set

$$Xv_{3i} = v_{3i+1}, \quad Xv_{3i+2} = v_{3i+3}, \quad Yv_{3i+2} = v_{3i+1},$$

with indices modulo $3m$ and all other values zero. Again mixed products and cubes vanish. An Euler operator would yield $2m - m = m = 0$ by the same telescoping calculation, impossible in characteristic zero. If a bounded finite-support complex pushed to this module in degree zero, exactness would make its zeroth cohomology a module lift, a contradiction. □

The source's derived result is a *precovering* in the stated bounded categories [74, Theorem 4.28]; density remains an additional condition. Neither this result nor the GLS equivariant classification [75] identifies our completed order with a GLS algebra. Graded surface/Fukaya models [76] likewise require their own dg category, stops and grading. These categorical tools do not construct the independent positive arithmetic determinant.

8 Verification and research frontier

The two nodal/invariant Lean files construct the node as an actual subring of $k[[X]] \times k[[X]]$ with equal constants, prove its annihilators, regular common parameter, augmentation, alternating projective resolution and a bijective pair of power-series coordinates. A second file proves CK1 collapse under the stated permutation/coverage hypotheses, determinant base change, actual 2×2 symmetric-square functoriality and relation preservation, a Schur character identity and the fixed-parameter lattice failure. These thirty printed theorem types have only standard Lean foundations.

They do not formalize general dessins, abstract Ext/AR/Schur–Weyl, C* exactness, Fredholm determinants, arithmetic descent or GRH. Earlier formal layers remain preserved with their literal scopes.

Exact-rational diagnostics independently check the displayed dessins, path identities, Schur and Young examples and the nine-object grid. These finite checks supplement the proofs, rather than prove infinite spectral or arithmetic assertions. The next arithmetic target is an exact descent/Frobenius-compatible construction on the labelled order/Fock data, with a determinant-preserving positive comparison. The local Schur kernel and the simple degree spectrum alone do not supply it. All source clones and earlier research are retained; the earlier general focused PDFs are archived at the preceding commit.

Ten further declarations in `DessinPullbackComponents.lean` check actual matrix-unit identities, nonmultiplicativity of raw diagonal extraction, impossibility of a unital two-state matrix residue into a nonzero commutative ring, multiplicativity on diagonal fibers, and genuine equal-image pullback subring/projection components. The analytic suite then contained187printed declarations at unchanged pins. The circle-function C* model, functional calculus, generation, graph uniqueness and K-theory remain prose/external-theorem arguments, not hidden assumptions of these component checks.

Sixteen additional declarations in `ThetaJacobi.lean` check genuine finite-field MulChar Jacobi conjugation, norm and rescaling with explicit nontriviality hypotheses; cubic–quadratic noncancellation; the cyclotomic multiplication determinant and positive Gram identity; and actual theta cellular boundary, homology-inclusion, restriction and rational chain-map matrices. The full analytic suite now contains203declarations. The point-count proof, deck/Kummer comparison, cohomological trace interpretation and global arithmetic comparison are not thereby fully formalized.

The final three `ThetaJacobi.lean` declarations verify the actual conjugation matrix identities and the real two-dimensional inert-prime determinant obstruction. They do not assume any unknown zero location or exclude complex/enlarged spectral realizations.

`ThetaProjectiveMultiplicity.lean` adds five declarations. Arbitrary upper-block matrices with six identical diagonal blocks have the sixth-power determinant and characteristic-factor formulas. A literal power-series constant-coefficient map specializes matrix determinants, for each scalar parameter; the coefficientwise polynomial/order interface remains prose. A nonzero quadratic cannot be a rational-function sixth power, proved by polynomial degrees. Three genuine power series with product X cannot have the cyclically equivalent residue-vanishing conditions forced by invertible rank-one arrow intertwiners. The derivation of the path blocks from the full theta order, projective decomposition, derived branch functor and deck intertwiners is not fully formalized. The complete analytic suite now has208printed declarations; no hidden arithmetic positivity assumption occurs.

`ThetaTameDescent.lean` adds twelve printed component declarations (and one auxiliary polynomial lemma). They use genuine formal-power-series rescaling and its coefficient operator, proving the coefficient action, invariant-coefficient vanishing and the tame operator's cube. Actual three-state matrices satisfy the clock/rotation commutator, comparison determinant and closed-fiber equations, weighted-cycle cube and full characteristic polynomial, noncommutation of a nonconstant rotation polynomial, trivial whole-column determinant actions, and the same-operator comparison for vC. The classification of all fixed series by residues, full order action and hereditary invariant algebra, stack interpretation, valuation/rank theorem, general coefficient-functor filtration and GRH are not thereby fully formalized. No unknown positivity premise is assumed.

9 Why an absolute coprime pair-square estimate cannot close the short rows

Keep the original sextic family

$$M_u(D) = D^{-1/2} \sum_n \mu(n)\nu(n) W(Nn/D)\chi_n(u), \qquad \chi_n(u) = (u/n)_6,$$

with squarefree primary column ideals outside the fixed excluded set, their full annular coefficients (fix W supported in $[1,2]$), and all nonzero Eisenstein element rows $Nu \le H$. The cubic symbol is χ_n^2, not χ_n. The following classical frame inequality [80] gives a new obstruction to one proposed estimate in this arithmetic program.

Proposition 9.1 (Coprime pair-square obstruction)**.** *Let A be an R-by-N complex matrix with $|A_{ui}| = 1$, and put $K_{ij} = \sum_u A_{ui}\overline{A_{uj}}$. Then*

$$\sum_{i \ne j} |K_{ij}|^2 \ge NR(N-R).$$

For the original character kernels, let $B_1(H,D)$ be the sum of $|K_H(n,m)|^2$ over all distinct coprime squarefree column ideals in $[D,2D]$. At $H = D^{3/4}$,

$$B_1(H,D) \gg \frac{D^{11/4}}{(\log D)^2}.$$

In particular the proposed sufficient input $B_1(H,D) \ll H^2 D^{\varepsilon}$ fails for each fixed $\varepsilon < 5/4$. This does not refute the signed Möbius mean square.

Proof. Set $F_{uv} = \sum_i A_{ui}\overline{A_{vi}}$. Reordering four finite sums proves $\sum_{i,j} |K_{ij}|^2 = \sum_{u,v} |F_{uv}|^2$. The row diagonal contributes RN^2; the column diagonal contributes NR^2. Removing the latter and retaining the former gives the inequality.

Restrict the column pairs in the nonnegative sum B_1 to distinct prime ideals p, q of norm in $[D,2D]$, outside the same fixed set. If $H < D$, no such prime divides a nonzero row element: divisibility would imply $Nu \ge Np$. Thus all original zero masks are exactly one on this ball, $|\chi_p(u)| = 1$, and the residual primitive finite character conductor is pq, of norm between D^2 and $4D^2$. All these pairs have common divisor1. The lattice count gives $R = (2\pi/\sqrt{3})H + O(\sqrt{H}+1)$. The classical unconditional fixed-field prime-ideal theorem gives $N \sim D/\log D$; this is an external analytic input, not a GRH hypothesis. At the stated endpoint $R = o(N)$, and the bound is therefore $(1-o(1))N^2R \gg D^{11/4}/(\log D)^2$. □

To see the exact failed step, write $c_n = \mu(n)\nu(n)W(Nn/D)$ and $C_D = \sum_n |c_n|^2 = O(D)$. Cauchy over coprime pairs bounds the absolute signed contribution by $D^{-1}C_D B_1^{1/2}$. The hoped-for pair-square upper bound would have closed this particular route, but is false. A large absolute pair-square mass is not a lower bound on the quadratic form with the fixed c_n. Nor is the prime part a positive summand of the full Möbius energy: prime–composite cross terms can cancel. The original low-gcd estimate and the positive arithmetic determinant bridge remain open.

10 A signed Möbius decomposition and its termwise obstruction

Keep the original squarefree good-ideal family, its fixed ν, its full $W(Nn/D)$ and every zero-extended sextic symbol. In ideal Dirichlet convolution, write $\zeta_0(n) = 1$ for every good ideal and let δ be

the convolution unit. For $1 \le U < D$ put $a_U = \mu \mathbf{1}_{Nn \le U}$ and $b_U = \mu - a_U$. The classical Vaughan algebra gives

$$\mu = 2a_U - a_U * a_U * \zeta_0 + b_U * b_U * \zeta_0. \tag{1}$$

Indeed $\mu * \zeta_0 = \delta$, and expansion of the last product proves the identity without an analytic assumption. This is a classical decomposition, not a new cancellation theorem; compare [87, Lemma4.1].

On squarefree n all factors in $n = dc = abc$ are pairwise coprime. Define the actual, nonarbitrary coefficient functions

$$\beta_U(d) = \sum_{\substack{ab=d \\ Na, Nb \le U}} \mu(a)\mu(b), \qquad g_U(s) = \sum_{\substack{b \mid s \\ Nb > U}} \mu(b).$$

For squarefree d, $\beta_U(d)$ is $\mu(d)$ times the number of the displayed ordered factorizations. For $Ns > U$, $g_U(s) = -\sum_{b \mid s,\, Nb \le U} \mu(b)$. The two regrouped pieces are

$$I_u = D^{-1/2} \sum_{\substack{d,c \text{ squarefree} \\ (d,c)=1}} \beta_U(d)\nu(dc)\chi_d(u)\chi_c(u)W(NdNc/D),$$

$$II_u = D^{-1/2} \sum_{\substack{a,s \text{ squarefree} \\ Na, Ns > U,\, (a,s)=1}} \mu(a) g_U(s)\nu(as)\chi_a(u)\chi_s(u)W(NaNs/D).$$

Then $M_u = -I_u + II_u$. All ideals here remain outside the same fixed excluded set. The short $2a_U$ term vanishes because $U < D$ and W is supported in $[1, 2]$. No phase or coprimality mask was removed.

Proposition 10.1 (An all-cutoff separate-piece bound is false)**.** *Take the allowed principal ν, $U = 1$, and a fixed nonnegative smooth W supported in $[1, 2]$, bounded below by a positive constant on a closed subinterval of $(1, 2)$ of positive length. For each of the two pieces,*

$$\sum_{0 < Nu \le H} |I_u|^2 \gg_W \frac{D}{\log^2 D}, \qquad \sum_{0 < Nu \le H} |II_u|^2 \gg_W \frac{D}{\log^2 D} \quad (H \ge 1).$$

Consequently a proposed bound HD^{ε} for each piece at $H = D^{3/4}$ fails for $\varepsilon < 1/4$. In the exact original notation $H(HD)^{\varepsilon}$, it fails for $\varepsilon < 1/7$. This refutes an all-cutoff termwise strengthening, not the original signed mean square or a specified moving-cutoff argument.

Proof. At $U = 1$, $a_U = \delta$, so $a_U * a_U * \zeta_0 = \zeta_0$ and $b_U * b_U * \zeta_0 = \mu + \zeta_0 - 2\delta$. On the annulus the two coefficients are 1 and $1 + \mu(n)$, respectively. On squarefree ideals the latter is zero or two. At the actual unit row $u = 1$, every original character equals one. A fixed-field prime-ideal theorem gives $\gg D/\log D$ good prime columns in the subinterval, so $I_1 \gg_W \sqrt{D}/\log D$. Fix a good prime ideal q_0. Products $q_0 p$ with varying distinct good primes in the rescaled subinterval have $\mu(q_0 p) = 1$. Their nonnegative contributions give $II_1 \gg_W \sqrt{D}/\log D$. Keeping that one row in each nonnegative energy proves the result. The prime-ideal theorem is a classical unconditional external input, with the field and q_0 fixed. □

The factorization does not shorten the primitive row conductor: for squarefree coprime a, s it is as, not a or s separately. For two resulting columns n, n' with common factor h, the oscillatory conductor is nn'/h^2 and a zero-preserving period is nn'/h. The signed low-gcd estimate and the independent positive arithmetic determinant comparison therefore remain open.

11 A full-mask Hecke expansion for the type-I piece

We now continue the signed decomposition through its actual squarefree inner sum. Fix $K = \mathbb{Q}(\sqrt{-3})$, the finite-order Hecke character ν, and the same finite excluded prime set S, containing primes above 6 and the conductor of ν. For $u \in \mathcal{O}_K \setminus \{0\}$, let ψ_u be the primitive Hecke character induced by $\nu(n)(u/n)_6$ away from $S \cup \operatorname{supp}(u)$. It is defined before, and independently of, the type-I divisor d. The abelian Kummer extension $K(u^{1/6})/K$ constructs the sextic factor; multiplying by ν gives the stated character. This uses classical abelian reciprocity, not general Artin holomorphy. Write $\psi_{u,2}$ for the primitive character inducing its square and $\mathfrak{f}_u, \mathfrak{f}_{u,2}$ for the two primitive conductors. For a squarefree good ideal d put

$$T = T(u,d) = S \cup \operatorname{supp}(u) \cup \operatorname{supp}(d), \qquad R_T = \prod_{p \in T} Np, \quad Q_u = |D_K| N\mathfrak{f}_u.$$

All primes in T are deleted even if absent from either primitive conductor.

Theorem 11.1 (Restoration and a uniform scalar contour bound)**.** *For $\Re s > 1$ the original type-I inner Dirichlet series is*

$$F_{u,d}(s) = \sum_{\substack{c \text{ squarefree} \\ (c,T)=1}} \frac{\psi_u(c)}{(Nc)^s} = \frac{L(s,\psi_u)}{L(2s,\psi_{u,2})}\, C_T(s), \tag{2}$$

$$C_T(s) = \frac{\displaystyle\prod_{p \in T,\ p \nmid \mathfrak{f}_u} (1 - \psi_u(p)(Np)^{-s})}{\displaystyle\prod_{p \in T,\ p \nmid \mathfrak{f}_{u,2}} (1 - \psi_{u,2}(p)(Np)^{-2s})}. \tag{3}$$

It continues meromorphically to $\Re s > 1/2$, with only a possible simple pole at $s = 1$ when $\psi_u = 1$. In that case its residue is

$$\rho_T = \frac{\operatorname{res}_{s=1} \zeta_K(s)}{\zeta_K(2)} \prod_{p \in T} (1 + (Np)^{-1})^{-1}.$$

Fix $0 < \delta < 1/2$, $\varepsilon > 0$ and $W \in C_c^\infty((0,\infty))$ supported in $[1,2]$. Set

$$\mathcal{M}_4(W) = \sum_{j=0}^{4} \sup_x |(x\partial_x)^j W(x)|.$$

Uniformly for $Y \geq 1$, u and d,

$$\sum_{\substack{c \text{ squarefree} \\ (c,T)=1}} \psi_u(c) W(Nc/Y) = \mathbf{1}_{\psi_u = 1} \rho_T Y \widehat{W}(1) + O_{\nu,S,\delta,\varepsilon}\Big(\mathcal{M}_4(W) Y^{1/2+\delta} Q_u^{1/4-\delta/2+\varepsilon} R_T^{\varepsilon} \Big). \tag{4}$$

Here $\widehat{W}(s) = \int_0^\infty W(x) x^{s-1}\, dx$. The continuation and functional equation of degree-one Hecke functions are classical external inputs; the contour argument is given below.

At a good prime with $v_p(u) = 3$, ψ_u is ramified while its square is unramified; the second product in (3) must restore a factor invisible to squaring the zero-extended primitive value. At $v_p(u) = 6$

both characters can forget the prime, but the original row still vanishes on columns divisible by it. In particular, for principal ν, $u=\pi^3$ and $u=\pi^6$ at $N\pi=7$, the deleted local factors are

	primitive ratio at $s=1$	restoration	full masked factor
$v_p(u)=3$	48/49	49/48	1
$v_p(u)=6$	8/7	7/8	1

These are exact local calculations, not numerical evidence about zeros.

For $1\le U$, $U^2\le D$, insert $Y=D/Nd$ into (4) and retain the actual $\beta_U(d)$ and outer $\nu(d)\chi_d(u)$. Classical tame Kummer conductors give $Q_u\ll_\nu Nu$. Thus the normalized I_u equals its displayed principal-residue sum plus an error E_u with

$$|E_u|\ll_{\nu,S,\delta,\varepsilon}\mathcal{M}_4(W)D^{\delta}U^{1-2\delta+\varepsilon}(Nu)^{1/4-\delta/2+\varepsilon}. \tag{5}$$

The resulting elementary energy envelope is only

$$\sum_{0<Nu\le H}|E_u|^2\ll\mathcal{M}_4(W)^2D^{2\delta}U^{2-4\delta+\varepsilon}H^{3/2-\delta+\varepsilon}. \tag{6}$$

It does not prove the short-row target of order H up to small losses. This is an insufficient upper bound, not a lower bound on the actual error or a counterexample to coefficient-sensitive averaging. The original signed mean square and positive determinant bridge stay open.

12 Actual renormalization retains the effective action

A compression score does not specify an RG transformation. Following the real-space formulation of Wilson and Kogut [81], we specify the eliminated variables, effective action, normalization and scale map. Our exact finite decimation is distinct from an approximate tensor-network truncation such as Levin–Nave TRG [82].

Definition 12.1 (Finite classical RG step)**.** Let the fine configuration be $(s,t)\in S\times E$, with finite nonempty sets S,E, and dimensionless log Boltzmann weight $V(s,t)=-\beta H(s,t)$. Define

$$(\mathcal{R}V)(s)=\log\sum_{t\in E}e^{V(s,t)},\qquad Z(V)=\sum_{s,t}e^{V(s,t)}.$$

The full action includes its additive constant. A truncation or a learned approximation is a further operation with its own error, not this identity.

Proposition 12.2 (Partition, observables and composition)**.** *One has $Z(\mathcal{R}V)=Z(V)$, and the normalized expectation of every observable depending only on s agrees in the two descriptions. Eliminating E_1 and then E_2 equals eliminating $E_1\times E_2$ at once, with the same tensor labels and inverse temperature.*

Proof. Exponentiating the definition gives $e^{\mathcal{R}V(s)}=\sum_t e^{V(s,t)}$. Sum over s, optionally multiplying by an observable $O(s)$, and divide both sides by the same strictly positive Z. Reordering finite sums proves the composition statement. No projection onto pair interactions is used. □

For an even homogeneous periodic Ising chain, write the transfer matrix as $T_{ab}=\exp(C+Kab+h(a+b)/2)$, $a,b\in\{-1,1\}$. Its partition function is $\operatorname{tr}T^N$. Summing alternate spins replaces T by T^2 and N by $N/2$; the physical spacing doubles. Writing $P=T^2$, its positive entries determine

$$K'=\tfrac14\log\frac{P_{++}P_{--}}{P_{+-}^2},\quad h'=\tfrac12\log\frac{P_{++}}{P_{--}},\quad C'=\tfrac14\log(P_{++}P_{--}P_{+-}^2).$$

These three parameters reproduce every entry of P, so no normalization is lost. At $h = 0$, $\tanh K' = \tanh^2 K$ and the eigenvalue-ratio correlation length of the homogeneous extension in blocked units halves. Its product with the spacing stays fixed. The fixed points in $t = \tanh K$ are $t = 0, 1$; the latter is zero temperature, not a positive-temperature transition. Odd dessin examples below use the full-action definition, not this even-chain shortcut.

12.1 Genuine dessin incidence models

For odd m, the clean dessin $y^2 = x^m - 1$, $\beta = x^m$ has two black vertices, m white vertices and one face; its genus is $(m-1)/2$. Its incidence graph is $K_{2,m}$ with $2m$ clean edges. Its infinite locally gentle order quiver instead has m vertices and completed DVR-rank $2m^2$. We place physical spins $B_+, B_-, s_0, \ldots, s_{m-1}$ on the incidence vertices; this is a specified finite spin model, not a faithful representation of that infinite algebra or order. The abstract incidence graph alone does not recover the ribbon embedding or genus.

With $\eta_j = (-1)^j$ and fixed nonuniform fields h_j, choose

$$V = J_+ B_+ \sum_j s_j + J_- B_- \sum_j \eta_j s_j + b B_+ - \tfrac{3}{5} b B_- + \sum_j h_j s_j.$$

Summing the two black spins gives the *entire* effective action

$$V'(s) = \sum_j h_j s_j + \log\left[2\cosh\left(b + J_+ \sum_j s_j\right)\right] + \log\left[2\cosh\left(-\tfrac{3}{5} b + J_- \sum_j \eta_j s_j\right)\right].$$

Every one of the $2m$ physical edges is present, although its couplings belong to a tied three-parameter family. The Walsh expansion $V'(s) = \sum_{A \subseteq \{0,\ldots,m-1\}} c_A \prod_{j \in A} s_j$, with $c_A = 2^{-m} \sum_s V'(s) \prod_{j \in A} s_j$, is unique by Boolean orthogonality. It retains the constant $c_\varnothing$ and genuinely generated higher-body interactions. Our genus-two and genus-four examples use $m = 5, 9$ and continue by exact nested marginalization. Dropping all terms of degree greater than two is displayed as a measured approximation.

12.2 Quantum thermal effective Hamiltonians

For a finite Hermitian Hamiltonian H on nonzero finite-dimensional $\mathcal{H}_S \otimes \mathcal{H}_E$ and $\beta > 0$, define

$$W' = \operatorname{Tr}_E e^{-\beta H}, \qquad H_{\text{eff}}(\beta) = -\beta^{-1} \log W'.$$

The matrix W' is strictly positive: for $v \neq 0$, sum the strictly positive quadratic forms of $v \otimes e_j$ over an orthonormal basis of $\mathcal{H}_E$. Its Hermitian spectral logarithm is therefore defined. The identity $e^{-\beta H_{\text{eff}}} = W'$ proves partition and retained *equal-time* observable preservation by the partial-trace property. Repeated traces compose. The full Pauli expansion, including its identity coefficient, is retained. It is generally nonlocal and depends on β; this statement does not preserve all fine real-time dynamics or supply a local zero-temperature/thermodynamic RG fixed point.

`FiniteRG.lean` checks nine actual finite scalar marginal, exponential-action, partition, observable, composition, error-sandwich and transfer-trace statements. The quantum spectral-log and complete thermodynamic/field-theory interfaces are not formalized by these lemmas. These physics constructions give no arithmetic determinant comparison or new GRH conclusion.

13 Linear relations and functorial branch filtrations

The original linear-relations chapter mixes graphs, partial relations and stable kernels. We keep these different objects separate. For a field k, a linear relation is a subspace $C \subseteq V \oplus V$; its inverse swaps coordinates and $DC = \{(x,z) : \exists y, (x,y) \in C, (y,z) \in D\}$. The full graph $\Gamma_T = \{(x, Tx) : x \in V\}$ always has dimension $\dim V$, including pairs with zero output. In particular $\Gamma_T^n(0) = 0$, whereas $(\Gamma_T^{-1})^n(0) = \ker T^n$. Thus the source's $\kappa(\Gamma_T) = \bigcup_n \Gamma_T^n(0)$ is zero, not the ordinary stable kernel.

For odd $m \geq 3$, use the actual clean dessin of $y^2 = x^m - 1$, $\beta = x^m$, on the compact smooth curve, in characteristic zero. Its genus is $(m-1)/2$; its infinite locally gentle algebra A_m has vertices $i \in \mathbb{Z}/m$, arrows $a_i, b_i : i \to i-1$, and mixed consecutive branch products zero. Below paths act in their direction of travel on left representations over any coefficient field k. Let $\mathcal{E}_m$ be the full category of finite-dimensional representations with every $b_i = 0$ and a_i invertible for $i \neq 0$.

Theorem 13.1 (A concrete relation category on an infinite dessin algebra)**.** *The category $\mathcal{E}_m$ is equivalent to finite-dimensional pairs (V,T) with intertwining maps. A normal form $N_m(T)$ puts V at each vertex, $a_0 = T$, and all other $a_i = 1$. For $n = \dim V$,*

$$V = K(T) \oplus I(T), \qquad K(T) = \ker T^n, \quad I(T) = \operatorname{im} T^n.$$

These summands and their kernel/image filtrations are functorial. The first restriction is nilpotent and the second invertible. The central full-turn parameter u acts as T. A finite $N_m(T)$ extends continuously to the u-adically completed order precisely when T is nilpotent. Such modules are torsion, not nonzero order lattices.

Proof. Transport all vertex spaces to vertex zero along the invertible arrows; the remaining map is the full-turn monodromy T. Arrow-compatible maps become exactly $hT = T'h$. Finite kernel/image chains stabilize by step n. If $v = T^n w$ and $T^n v = 0$, then $w \in \ker T^{2n} = \ker T^n$, hence $v = 0$; rank-nullity proves the direct sum. Stability gives the two restrictions, and $hT^r = T'^r h$ proves functoriality. Continuity on a finite-dimensional discrete space means some u^r kills it. Conversely nilpotence makes every formal series act by a finite sum. This proves the completion assertion. □

Over an algebraically closed k, the indecomposable branch pairs are $J_r(\lambda)$. The value $\lambda = 1$ is permitted for a polynomial band; it is not a forbidden deformation parameter. The nilpotent blocks are strings, but are not all rigid: $N_m(J_2(0))$ is a nonsplit self-extension of $N_m(J_1(0))$. The Hom dimension between two such Jordan pairs is $\min(r,s)$ for equal eigenvalues and zero otherwise. These assertions concern this specified full subcategory, not all infinite modules.

If L transports once along the a-arrows on $\bigoplus_i V$, then

$$\det(1 - zL) = \det(1 - z^m T). \tag{7}$$

Eliminating the identity arrow blocks leaves the final monodromy block; the eliminated identity blocks have determinant one. For $J_r(\lambda)$ the answer is $(1 - \lambda z^m)^r$. Nilpotent strings therefore contribute one, and arbitrary bands have no forced unit-circle zero location. This determinant is an actual cyclic-quiver invariant; it is not the missing arithmetic positive determinant comparison.

14 A kernel-checked extension over the infinite dessin algebra

The preceding branch example can be realized on every element of the infinite algebra, rather than verified only on sampled arrow matrices. Use the complex genus-two clean dessin $y^2 = x^5 - 1$,

$\beta = x^5$, and any coefficient field k. Put $H \subseteq M_5(k[t])$ equal to the matrices whose reduction modulo t is upper triangular. Let

$$B = H \times_{k^5} H,$$

where each map to k^5 takes the constant diagonal. Equivalently, B is generated inside $M_5(k[t])^2$ by

$$e_i = (E_{ii}, E_{ii}), \quad a_i = (w_i E_{i-1,i}, 0), \quad b_i = (0, w_i E_{i-1,i}), \qquad w_0 = t, \quad w_i = 1 \ (i \neq 0).$$

Indices are modulo five. Matrix multiplication acts on left column modules. The path-normal-form proof below identifies B with A_5; this is a polynomial algebra, not an arrow-truncated finite quotient.

Theorem 14.1 (An ambient module extension)**.** *Let $D = k[\epsilon]/(\epsilon^2)$, $M_1 = k^5$ and $M_2 = D^5$. For $(C(t), D'(t)) \in B$, act on M_1 by $C(0)$ and on M_2 by $C(\epsilon)$. These are unital B-modules, and*

$$0 \longrightarrow M_1 \xrightarrow{i} M_2 \xrightarrow{p} M_1 \longrightarrow 0, \qquad i(x) = \epsilon x, \quad p(x + \epsilon y) = x,$$

is exact and nonsplit. Moreover B is infinite dimensional over k. The assertions hold in every characteristic, including two.

Proof. Evaluation of polynomials gives both actions. Multiplying $C(\epsilon) = C_0 + \epsilon C_1$ proves B-linearity of i, p; their kernel and image follow coordinatewise. If s were a B-linear section, the constant coefficient of $(se_0)_0$ would be one. But a_0 kills M_1, while $(a_0 se_0)_4 = \epsilon(se_0)_0$ has nonzero ϵ coefficient, a contradiction. Finally the permitted turn $c = a_1a_2a_3a_4a_0$ has first component tE_{00}. The unital algebra map $k[t] \to B$, $f \mapsto f(c)$, is injective: its $(0,0)$ first-component entry is $f(t)$. Thus B cannot be finite dimensional. □

In the preceding notation these modules are $N_5(J_1(0))$ and $N_5(J_2(0))$. They are finite torsion modules over an infinite algebra; they are not nonzero lattices over its completed surface order. The nonsplit sequence defines a nonzero Yoneda extension class in the ambient module category. No assertion that it is almost split in that entire category is needed or inferred. In particular it does not supply an arithmetic spectral determinant or a zero-location theorem.

Theorem 14.2 (Unbounded normal-image faithfulness)**.** *In $M_5(k[t])^2$, index the stationary images by i and the positive images by (b, i, j, q), where $b \in \{a, b\}$, $0 \leq i, j < 5$, $q \geq 0$ and $i < j$ or $q > 0$. The positive image is $t^q E_{ij}$ in factor b, with zero in the other factor. Its path length is $5q + j - i > 0$. All these images, including the shared stationary images, are linearly independent over every field. Every finite normal expansion has uniquely recoverable coefficients, without a bound on degree or path length.*

Proof. Read a stationary coefficient from the first factor's constant diagonal; read a positive coefficient from the indicated factor, entry and degree. The index restriction ensures that no positive image contributes to a constant diagonal. These linear selectors are biorthogonal to the images, so applying each selector to a finite expansion recovers its coefficient. □

This theorem is now kernel checked for actual polynomial matrix pairs. The membership and spanning steps are supplied by the next theorem. Comparison with a separately constructed path quotient remains a distinct algebraic formalization task.

Theorem 14.3 (Normal coordinates in the generated algebra)**.** *Let $A \subseteq M_5(k[t])^2$ be the algebra generated by the displayed vertices and arrows, and let $\mathcal{K}$ be the normal index set of Theorem 14.2. Every normal image belongs to A. The actual A-valued finite expansion*

$$E : k^{(\mathcal{K})} \longrightarrow A, \qquad c \longmapsto \sum_{x \in \mathcal{K}} c_x v_x$$

is a linear isomorphism, whose inverse is $(E^{-1}a)_x = \lambda_x(a)$*. Thus the normal images form a basis of the infinite generated algebra, not merely an independent family in its ambient matrix space.*

Proof. Write $P_b(i,j)$ for the shortest positive branch walk from j to i, using a full turn when $i = j$. Its image is $t^d E_{ij}$ in branch b, where $d = 0$ if $i < j$ and $d = 1$ otherwise. The actual arrow product $P_b(i,i)^n P_b(i,j)$ has image $t^{n+d}E_{ij}$ for every $n \geq 0$. Admissibility gives $q \geq d$, so $n = q - d$ proves membership.

The normal span contains the generators and $1 = \sum e_i$ and is multiplicatively closed: stationary factors select endpoints; positive factors either multiply to zero or to $t^{q+r}E_{il}$ in the same branch. When $q + r = 0$, the two strict upper inequalities imply $i < l$, so this output slot is still permitted. Adjoin induction gives spanning of A. The previous coefficient selectors prove uniqueness and the inverse formula. □

All membership, spanning and finite-expansion assertions in this theorem are now Lean checked over arbitrary fields. The formal linear equivalence is not asserted to be an algebra map from an independently defined path quotient, nor a statement about completed orders, derived categories or GRH.

15 Learned fields, width limits and quantum geometry

Learned functions enter this program as candidate generators, physical surrogates and control laws. A network may suggest an arithmetic metric, operator or gluing datum; a successful fit does not establish its positivity, analytic domain or identification with a completed L-function. Those are independent mathematical obligations. This section gives explicit interfaces and executed examples rather than treating "quantum", "tropical" or "infinite width" as a substitute for that comparison.

Proposition 15.1 (A specified quantum-feature width limit)**.** *Let* $x \in \mathbb{R}^d$*,* $w \sim N(0, I_d)$ *and* $b \sim \mathrm{Unif}[0, 2\pi]$ *be independent, and set* $\phi_{w,b}(x) = \sqrt{2}\langle Z\rangle_{R_Y(w\cdot x+b)|0\rangle} = \sqrt{2}\cos(w \cdot x + b)$*. For independent features,*

$$K_m(x,y) = m^{-1}\sum_{r=1}^{m} \phi_r(x)\phi_r(y) \longrightarrow K(x,y) = e^{-\|x-y\|^2/2}$$

almost surely on every specified finite set of pairs. Every K_m *is a positive semidefinite Gram kernel. For a readout* $f_a(x) = m^{-1/2}\sum_r a_r\phi_r(x)$ *with only a trained, its tangent kernel is exactly* K_m*. These statements do not assert the same limit for a jointly trained finite quantum transformer.*

Proof. Uniform-phase integration gives $\mathbb{E}_b[2\cos(u + b)\cos(v + b)] = \cos(u - v)$. The Gaussian characteristic function then gives K. Products are bounded by two, so the strong law applies to each pair and hence any finite union. For any finite coefficient vector c, the Gram quadratic form is $m^{-1}\sum_r |\sum_i c_i\phi_r(x_i)|^2 \geq 0$. Differentiating with respect to a_r gives the final assertion. □

The frozen-kernel squared-loss flow has residual $r(t) = \exp(-K_m t)r(0)$ when the loss normalization is absorbed into time. Null directions do not decay. A complex feature space instead uses the Hermitian kernel $\mathbb{E}[\phi(x)\overline{\phi(y)}]$. A finite coefficient field has neither this Gaussian law nor an ordered positive Hilbert metric; changing algebraic coefficients is a separate operation.

Proposition 15.2 (Tropical limit and a cancellation boundary)**.** *For real* $s_1, \ldots, s_n$ *and* $\tau > 0$*,*

$$0 \leq \tau \log \sum_j e^{s_j/\tau} - \max_j s_j \leq \tau \log n.$$

An attention maximum therefore has a controlled log-sum-exp approximation. It does not by itself define a tropicalization of a variety or an HMS functor. In a valued coefficient field, valuation of a sum can increase by cancellation.

Proof. After subtracting $M = \max s_j$, the exponential sum lies between one and n. For the second statement, both $1+T$ and $-1+T$ have valuation zero in $\mathbb{Q}((T))$, whereas their sum $2T$ has valuation one. In characteristic two that sum is zero. Thus replacing every addition by its tropical minimum without tracking leading coefficients loses information. □

Kontsevich–Soibelman constructions require valued analytic and integral-affine gluing data; appropriate gentle/Fukaya comparisons require specified categories, gradings, stops and smoothness hypotheses [92, 91, 93]. Learned coefficients can propose such data, but the compatibility equations and arithmetic comparison still need proofs.

Executed physical examples. On the actual clean genus-two dessin, put seven quantum spins on its $K_{2,5}$ incidence graph and compute a normalized Gibbs state. Exact white-spin partial trace gives a two-spin quantum RG marginal. Twelve RTX4090 runs train four-qubit-feature diffusion and flow-matching models for its smoothed Z-measurement law: two objectives, entangled/product circuit controls, and three seeds. They have 5,810 nominal parameters and 2,000 steps each. A separate four-qubit variational purification flow learns the full known density interpolation, with 48 controls. Its minimum fidelity on a 33-point grid is 0.999903; this is not a uniform theorem or universal noise inverse. The matched controls show no consistent entanglement advantage.

Actual cuStateVec/cuTensorNet execution also covers 7-, 11- and 23-qubit finite circuits on genus-two, genus-four and genus-ten dessin incidence graphs. The 23-qubit case has 8,388,608 amplitudes and 447 gates. These larger circuits are untrained; the small cases have independent NumPy comparisons, while the largest has normalization and inverse-circuit checks. The exact environment and all failed/preliminary records are preserved.

Geometric and hardware interfaces. For the contracted m-edge dessin, the graph fundamental group is F_{m-1}, and its universal graph cover is the m-regular tree. The theta case $m = 3$ has a three-regular cover, distinct from the four-regular Cayley graph of F_2. Neither is the coefficient field $\mathbb{F}_2$ or the universal cover of the filled compact surface. Explicit path modules over the infinite genus-two algebra give Hom, Ext and Tor computations; Frobenius pants tensors give exact TQFT gluing, with a heralded physical implementation. CHSH gives a finite game benchmark, distinct from the unrestricted correlation separation in MIP*=RE [99]. Three executable tropical hardware references and a source-pinned chip-design survey complement the circuits. They are arithmetic/control designs, not synthesized chips, validated CFT/string dualities or a proof of GRH.

16 Complementary physical datasets and checked dynamics

The learned program now has two additional, separately constructed corpora: 16,384 effective-material response records and22,951 molecular conformer/force records in their default configurations. Time samples are not counted as independent systems. Source-linked and dynamical configurations are additional. The active HolonForge/Aster and Mora-3 repositories were read only; no data, vocabulary, checkpoint, source enrollment or training process was changed.

The material corpus covers SSH, Kitaev, QWZ, Rice–Mele, Harper and actual clean-dessin classical RG models. The molecular corpus includes designed amides and peptides, with explicit-H MMFF94 calculations, geometric persistence, and method-labelled conformations. Forty-eight

existing Aster MP20 structures, 32 existing Mora PubChem identities and eight public PDB structures supply separately labelled anchors. Their new computed responses are not new experimental observations, DFT labels, binding affinities or CISS measurements.

All conformer, gauge, intervention and task descendants share a partition. Specified physical/chemical similarity edges are closed transitively before component-level stratification. Large components are kept intact even when 80/10/10 proportions or rare-class coverage become impossible. This gives a finite, threshold-relative separation statement; it cannot guarantee every kind of statistical independence or prevent overfitting. Source anchors are train only. Molecular anchors are also checked against every default heldout using the same chemical and applicable geometric thresholds; conflicting parent groups are excluded. Global exposure to the models' previous corpora is a separate evaluation gate.

For a number-conserving one-particle Hamiltonian H, write $P_{ij} = \langle c_j^\dagger c_i \rangle$ and $U(t) = e^{-itH}$. The data use $P(t) = U(t)P(0)U(t)^*$, occupations $n_i = P_{ii}$ and outgoing currents $J_{ij} = -2\,\mathrm{Im}(H_{ij}P_{ji})$. Then $\dot{n}_i = -\sum_j J_{ij}$, directly from $i\dot{P} = [H, P]$. Paired BdG evolution includes its independently computed condensate source; it is not falsely treated as electron-number-conserving normal transport. An exact interacting Hubbard dimer supplements Gaussian exchange correlations. Molecular nuclear dynamics uses MMFF94, with a separately labelled, uncalibrated effective π-orbital model for charge modulation.

Parquet configurations retain condition, target and evaluation roles, source versions/hashes, units, failures and audit evidence. The local dataset explorer shows actual structures, currents, time evolution, graph wavelets and checked circuit compilations. These computations support candidate discovery and model validation; the independent arithmetic determinant/zero-location bridge remains open. No new whole-chapter Lean formalization is asserted.

17 Molecular structure, complex transport and learned chemistry

The complementary-data laboratory now separates deposited protein structure, actual finite quantum transport and learned conformation generation. Covalent bonds use explicit molecular graphs or qualified PDB templates, with depth ordering and visible atom/bond scales. The ribbon view reads deposited `HELIX`/`SHEET` records; short synthetic peptide backbones remain unassigned rather than acquiring invented secondary structure.

Five additional finite propagation examples compare matched SSH/QWZ topological and trivial controls and an SSH domain wall. Complex phase, probability, current, coherence and selected eigenstates are distinct views. Analytic hydrogen volumes illustrate nodes, quantum beats and a phase vortex; they are not molecular electronic-structure calculations. Eight further CPU runs train classical and two-qubit hybrid score/flow models on periodic MMFF butane/pentane torsion spaces. All checkpoints, losses, controls and positive noise endpoints are explicit. These examples neither establish quantum advantage nor close the independent arithmetic bridge for GRH.

18 Learning objectives with explicit analytic guarantees

A useful learned contribution must distinguish approximation of known L-functions from proof of their zero locations. We propose holomorphic coefficient/readout learning, certified contour losses, arithmetic trace and functional-equation residuals, Schatten-controlled operator refinement, certificate search for the original signed moment, and exact-algebra representation constraints. Each has a different acceptance test.

For an analytic target on $|s-c| \le r$, let $|F| \le M_r$ there and choose $R < \rho < r$. A degree-N holomorphic readout trained on $m > N$ equally spaced points of the radius-ρ circle has the explicit uniform error budget

$$\|p_k - F\|_{|s-c|\le R} \le E_{\mathrm{opt}}(k) + E_{\mathrm{oracle}} + E_{\mathrm{alias}} + E_{\mathrm{Taylor}}.$$

The following proofs give all four terms, a primitive-Dirichlet arithmetic bound for M_r, and an infinite-feature kernel limit with its actual rates. Thus known analytic families can be approximated to arbitrary requested accuracy with independently certified evaluation error. This does not prove GRH. Uniform approximation by an already-real-zero determinant class is a different, potentially GRH-equivalent requirement; it must not be assumed through an alleged universal-approximation theorem.

19 Learning nonlocal strategies and checking their certificates

A two-player nonlocal game samples questions x, y and scores outputs a, b. The players may share a state, but communicate neither questions nor outcomes during play. Our finite tensor strategies have $p(a,b|x,y) = \mathrm{Tr}[\rho(P_a^x \otimes Q_b^y)]$. Offline joint optimization is allowed and is distinct from communication during the game. We train local measurement frames and a Schmidt angle using the exact Born objective, and replay actual checkpoints, probabilities, correlations, word moments and simulated shots in the interactive laboratory.

For $0 \le \alpha \le 2$ use the tilted score

$$S_\alpha = \alpha\langle A_0\rangle + \langle A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1\rangle.$$

Its classical maximum is $2+\alpha$; its quantum maximum is $\sqrt{8+2\alpha^2}$ [113]. At $\alpha = 0$ the uniform CHSH winning probability is $1/2 + S_0/8$. This conversion does not apply unchanged to the tilted objective. The three seeds for each of $\alpha = 0, 1, 3/2$ give nine trained runs. One $\alpha = 3/2$ run remains at the classical value 3.5; the other runs reach the stated quantum maxima numerically. This retained failure rules out a claim of universal convergence of the optimizer.

The magic-square task uses two Bell pairs, local four-dimensional joint measurements, row parity $+1$ and column parities $(+1, +1, -1)$. Its classical value is $8/9$ and its quantum value is 1, as proved below. Three further trained local-frame runs reach values within $3 \cdot 10^{-13}$ of one. All twelve runs are small CPU simulations with saved seeds and parameters, not hardware Bell experiments or evidence for quantum training advantage.

These examples belong to both the closure of finite tensor correlations C_{qa} and the commuting-operator set C_{qc}. The external theorem MIP* = RE separates these sets and implies the negative answer to Connes embedding [115]. We do not reproduce that theorem. Finite learned strategies, positive state Gram matrices and finite NPA constraints do not witness its separation.

20 Hubs, spokes and the topology retained by a dessin

The hubs-and-spokes construction in the HoTT book, Sections 6.6–6.8 [109], gives a useful language for attaching cells. For a loop $f : S^1 \to X$, form the homotopy pushout $X \cup_f CS^1$: add a fresh hub and paths from every $f(u)$ to that hub, continuously indexed by $u : S^1$. This attaches a disk, rather than merely a finite graph of radial edges. The book's Remark 6.7.1 explains why identifying the hub with an existing boundary point requires extra coherence; Remark 6.7.2 distinguishes propositional from judgmental computation.

For our dessins the useful distinction is geometric. A vertex star has a disk neighborhood with a specified cyclic ordering of its incident bands. Removing the disk center gives a punctured disk, which retains its peripheral loop. Removing only the vertex of a finite star instead leaves disconnected open rays. These operations cannot be interchanged. In a cellular dessin, filling a face imposes its attaching-word relation; leaving its center punctured retains a boundary component of the ribbon neighborhood.



disk
coherent circle of spokes

punctured disk
peripheral loop

The shaded regions are two-dimensional; the displayed finite spokes sample them. The annulus shown on the right is a deformation retract of the punctured disk, not a missing vertex in a one-dimensional graph.

Proposition 20.1 (Face filling changes the covering problem)**.** *Let a connected cellular dessin be a spine of an oriented genus-g ribbon surface with $n > 0$ boundary components. Its graph fundamental group is free of rank $2g + n - 1$. Capping all boundary components gives the presentation*

$$\left\langle a_1, b_1, \ldots, a_g, b_g \;\middle|\; \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle.$$

For the one-face clean dessins $K_{2,3}$ and $K_{2,5}$ used here, these free ranks are respectively 2 and 4, before capping.

Proof. A spanning-tree collapse of the connected spine leaves $E - V + 1$ circles. The ribbon surface retracts onto that spine and has Euler characteristic $2 - 2g - n = V - E$, proving the rank formula. Cutting a surface along a standard system of handle arcs gives the boundary word $\prod_i [a_i, b_i] \prod_j c_j$. Van Kampen therefore presents its group by these generators and this one relation. With a boundary present, eliminate c_n, leaving the stated free group. Capping attaches disks along the c_j, so van Kampen sets each $c_j = 1$, giving the closed-surface presentation. For our rotations on $K_{2,m}$, both black vertices have the same cyclic order on the m incident edges. Suppressing the bivalent white vertices gives dart rotations $(0 \cdots m-1)(m \cdots 2m-1)$ and involution $i \leftrightarrow i + m$. The face permutation is one $2m$-cycle for odd m. Thus $V = m + 2$, $E = 2m$, $F = 1$, and $g = (m-1)/2$, giving both examples. □

The once-punctured torus and closed torus both have $H_1 \cong \mathbb{Z}^2$, but their groups are F_2 and $\mathbb{Z}^2$; only the closed torus has $H_2 \cong \mathbb{Z}$. The analogous genus-two groups are F_4 and the four-generator surface group, with $H_1 \cong \mathbb{Z}^4$ in both cases. Indeed the standard closed-surface cellular complex has one vertex, $2g$ edges and one face. The face word is a product of commutators, so its abelianized boundary is zero. Removing the face leaves the wedge of circles. This computes the stated homology and also explains why H_1 alone misses the face relation. The torus group is abelian while F_2 is not; in higher genus the closed surface has nonzero H_2 while the graph does not.

Consequently the graph's universal cover is a tree, whereas the universal cover of the filled genus-two surface includes lifted faces and is a disk. Collapsing a spanning tree changes the combinatorial

cover graph; the Cayley tree of a free basis is not literally the original biregular dessin cover. This distinction matters when lifting strings, periodic paths and boundary conditions in the infinite locally gentle algebra.

The gluing language organizes local patches and their compatibility maps. It does not reconstruct the ribbon cyclic order, complex structure, Belyi map, quadratic zero relations, coefficient ring or polynomial/formal/operator completion from the homotopy type alone. These data are retained explicitly in the existing dessin and order constructions. Descent of modules, derived categories, Hom-control, AR meshes or Fukaya categories needs a separately defined functor and hypotheses; a homotopy pushout of underlying spaces is not such a theorem. Nor does it imply an arithmetic determinant identity.

The accompanying tool computes the same one-face genus-one/two/four dart families and draws local disk/annulus patches and their surface polygon presentations, peripheral loops and face words. These are exact finite combinatorial checks plus the topological proof above. HoTT is referenced as a conceptual foundation; no univalence or higher inductive-type axiom is added to the pinned Lean4 development. Its complete cell-attachment and categorical formalization remains an explicit task.

21 A finite, validated approximation result

Let $\chi_3 = (0, 1, -1)$, $\chi_4 = (0, 1, 0, -1)$ and $\chi_5 = (0, 1, -1, -1, 1)$, extended periodically, with parities $a = 1, 1, 0$. We approximate the completed functions

$$\Lambda_q(s) = (q/\pi)^{(s+a)/2}\Gamma((s + a)/2)L(s, \chi_q).$$

The primitive theta/Mellin identity is the explicit classical analytic input; no hypothesis about zeros is used. These three examples do not assert a uniform result for all conductors, general Hecke functions or Artin holomorphy.

Theorem 21.1 (Scope of the supplied finite certificate)**.** *For each $q = 3, 4, 5$, the supplied exact-rational calculation constructs a complex polynomial P_q of degree* 28 *satisfying*

$$\sup_{|s-1/2|\le 7/10} |P_q(s) - \Lambda_q(s)| < 5 \cdot 10^{-15}.$$

The computation includes enclosures of all 64 *true Fourier-circle nodes, completed-function sample error, analytic truncation, Fourier aliasing, the exact closed-form* 180*-step gradient-descent iterate and coefficient midpoint rounding. The Python implementation and informal analytic proof have independent checks; this is not a kernel-checked Lean theorem.*

Proof. Write $F(c+z) = \sum_{n\ge 0} a_n z^n$, with $c = 1/2$, $R = 7/10$, $\rho = 6/5$, $r = 12/5$. The independently recomputed rational theta majorants are strictly below $M_q = 4, 10, 6$, respectively, on $|z| \le r$. Use true roots $\zeta_j = e^{2\pi i j/64}$ and rational labels y_j with proved error $|y_j - F(c + \rho\zeta_j)| \le \eta$. Use the real Euclidean gradient on complex coefficient coordinates for

$$\mathcal{L}(b) = \frac{1}{128}\sum_{j=0}^{63}\left|\sum_{n=0}^{28} b_n\zeta_j^n - y_j\right|^2.$$

Discrete Fourier orthogonality gives the least-squares optimum $b_n^* = 64^{-1}\sum_j y_j\bar{\zeta}_j^n$, $0 \le n \le 28$, and gradient $\nabla\mathcal{L}(b) = b - b^*$. For step $1/4$ and zero initialization, the exact 180-step iterate is $b_n^{180} = (1 - (3/4)^{180})b_n^*$. The code evaluates this closed form; it does not claim an explicit180-update optimizer run.

Set $u = R/\rho = 7/12$, $v = R/r = 7/24$ and $w = (\rho/r)^{64} = 2^{-64}$. Let $\widehat{b}_n$ be an outward rectangle enclosing b_n^* and ϵ_n the sum of the coordinate halfwidths of the rectangle for b_n^{180}. Choose each actual polynomial coefficient as that rectangle's exact rational midpoint c_n. The normalized basis is explicit:

$$P_q(s) = \sum_{n=0}^{28} c_n \left(\frac{s - 1/2}{6/5}\right)^n.$$

Thus the exported coefficient with index n multiplies the normalized variable, rather than the raw power $(s-1/2)^n$. Its error on the compact disk is bounded by

$$E_{\text{opt}} = (3/4)^{180} \sum_{n=0}^{28} |\widehat{b}_n|_1 u^n, \qquad E_{\text{arithmetic}} = \sum_{n=0}^{28} \epsilon_n u^n,$$

$$E_{\text{oracle}} = \eta \frac{1-u^{29}}{1-u}, \qquad E_{\text{alias}} = M_q \frac{w}{1-w} \frac{1-v^{29}}{1-v},$$

$$E_{\text{Taylor}} = M_q \frac{v^{29}}{1-v}.$$

Here $|\widehat{b}_n|_1$ is the sum of the maximum absolute real and imaginary coordinates, a safe upper bound for every enclosed complex modulus. Cauchy's estimate $|a_n| \le M_q r^{-n}$ and the Fourier identity $b_n^* = a_n \rho^n + \sum_{\ell \ge 1} a_{n+64\ell} \rho^{n+64\ell}$ for exact labels prove the alias and Taylor bounds. Label perturbations contribute at most η per Fourier coefficient. The remaining two terms follow directly from the exact iterate and coefficient rectangles. Each term is rounded upward to a 180-bit dyadic rational; the reported total is the exact sum of those reported terms. The certified totals below are strictly below the asserted threshold. Detailed sample enclosures are proved in the extended edition and encoded with exact integer endpoints. □

Conductor	Maximum sample radius	Uniform polynomial bound
3	3.026×10^{-17}	1.786×10^{-15}
4	7.206×10^{-17}	4.455×10^{-15}
5	1.404×10^{-16}	2.906×10^{-15}

The displayed values are rounded upward; exact rational records establish the strict inequalities.

22 What this does and does not certify

The final JSON records exact rational centers, interval endpoints, source hashes and all five error terms. Float values are display summaries only. A high-precision comparison is an independent diagnostic, not the reason an interval contains its mathematical target. Arithmetic uses outward integer operations, and invalid float/order inputs are rejected, including through typed caches. Earlier pilots and failed validation probes are retained.

This is a fixed finite tolerance/configuration. Increasing accuracy needs new truncation and precision choices; the fixed $N = 11, T = 128$ interface is not an arbitrary-accuracy oracle. The broader approximation existence theorem is separate. The certificate says nothing about the location of zeros: P_q is not required to be real-rooted or a self-adjoint spectral determinant. The independent arithmetic determinant identity, positivity and original signed short-row/covariance estimates remain open. No GRH proof or complete new Lean4 formalization is claimed.

23 Requested-error readouts from preserved certified labels

The accepted final-v5 sample certificate is preserved unchanged. For each $q = 3, 4, 5$, it supplies $m = 64$ rational complex labels y_j, bounds $|y_j - \Lambda_q(c + \rho\zeta_j)| \le \eta_j$, and outward rectangles enclosing the true roots $\zeta_j = e^{2\pi ij/m}$, where $c = 1/2$, $\rho = 6/5$. The classical primitive theta/Mellin identity and accepted growth bounds $M_q = 4, 10, 6$ on $|s - c| \le r = 12/5$ remain the analytic inputs. This extension accepts an exact rational error request $\varepsilon > 0$ on the fixed compact $|s - c| \le R = 7/10$. It changes the finite readout, while retaining the certified sample errors and the 64-node alias budget. It is not an arbitrary-accuracy theta-sampling interface.

Theorem 23.1 (Executed rational readout with explicit refusal)**.** *For each supported conductor and rational request, the finite-data selection below either returns a normalized rational polynomial P and an executed, exactly replayable gradient-descent trace with*

$$\sup_{|s-c|\le R} |P(s) - \Lambda_q(s)| \le \varepsilon,$$

or refuses the request with the exact minimum base budget over all 64 allowed degrees. Refusal concerns attainability by this particular fixed-data majorant; it is not an impossibility theorem for approximation.

Proof. Let B_n be an outward rectangle enclosing $b_n^* = m^{-1} \sum_j y_j \bar{\zeta}_j^n$, $0 \le n < 64$, computed from the preserved root rectangles. Put $t_n = \mathrm{mid}(B_n)$ and let e_n be the sum of its coordinate halfwidths. These are exact rationals, and $|t_n - b_n^*| \le e_n$. For a selected $d < 64$ execute, using exact rational operations,

$$b_n^0 = 0, \qquad b_n^{k+1} = \tfrac{3}{4} b_n^k + \tfrac{1}{4} t_n, \qquad 0 \le n \le d.$$

This is genuine gradient descent with step $1/4$ for the explicitly specified real Euclidean objective $J(b) = \frac{1}{2} \sum_{n=0}^{d} |b_n - t_n|^2$, whose gradient is $b - t$. All iterates and exact objective values are retained. The output is literally

$$P_k(s) = \sum_{n=0}^{d} b_n^k \left(\frac{s - c}{\rho} \right)^n.$$

Set $u = R/\rho = 7/12$, $v = R/r = 7/24$, $w = (\rho/r)^{64} = 2^{-64}$, $S_d(x) = \sum_{n=0}^{d} x^n$ and $\eta = \max_j \eta_j$. The five budgets are

$$E_{\mathrm{opt}}(k) = \sum_{n=0}^{d} |b_n^k - t_n|_1 u^n, \qquad E_{\mathrm{oracle}} = \eta S_d(u),$$

$$E_{\mathrm{alias}} = M_q \frac{w}{1 - w} S_d(v), \qquad E_{\mathrm{Taylor}} = M_q \frac{v^{d+1}}{1 - v},$$

$$E_{\mathrm{arithmetic}} = \sum_{n=0}^{d} e_n u^n.$$

Here $|x + iy|_1 = |x| + |y|$. Cauchy's estimate and true-root Fourier orthogonality give the alias and Taylor terms; certified label errors give the oracle term. Triangle inequality applied to $b_n^k - b_n^* = (b_n^k - t_n) + (t_n - b_n^*)$ gives the other two. Thus their exact sum bounds the compact-uniform error.

Define $C_d = E_{\mathrm{oracle}} + E_{\mathrm{alias}} + E_{\mathrm{Taylor}} + E_{\mathrm{arithmetic}}$. Select the smallest $d < 64$ with $C_d < \varepsilon$; if none exists, report $\min_{d<64} C_d$ and refuse. For a selected degree, $E_{\mathrm{opt}}(k) = (3/4)^k \sum_n |t_n|_1 u^n \longrightarrow 0$. Therefore executing updates until $E_{\mathrm{opt}}(k) \le \varepsilon - C_d$ terminates after finitely many steps. The serialized coefficients, budgets, targets and trace use exact rational endpoints; displayed decimal approximations are not proof inputs. □

24 Requested-error theta samples for three fixed conductors

For the primitive characters $\chi_3 = (0, 1, -1)$, $\chi_4 = (0, 1, 0, -1)$, $\chi_5 = (0, 1, -1, -1, 1)$ with parities $a = 1, 1, 0$, retain the classical primitive theta/Mellin identity and root number one already stated above. Put $D = [-1, 2] + i[-4/3, 4/3]$ and $\delta = 3/q$. The following adaptive extension is implemented with independent interval and sample checks; it is an informal analytic theorem, not a Lean result.

Theorem 24.1 (Finite adaptive sample specification)**.** *For every supported conductor, exact rational $s \in D$, and rational $\eta > 0$, the choices below terminate and give a rational complex midpoint y and exact rational radius $R \le \eta$ with $|y - \Lambda_q(s)| \le R$. The same conclusion holds for a specified true circle node when its root and node enclosures are refined on every precision retry.*

Proof. For $x \ge 0$ define the exact rational bound $E(x) = 2^{-\lfloor x \rfloor}$. Since $e > 2$, $e^{-x} \le E(x)$; this bound is nonincreasing and tends to zero. Let $Q(x) = x^{-1} + 2x^{-2} + 2x^{-3}$ and $C_0 = 1$, $C_1 = (1 + \delta^{-1})/2$. For $N \ge 1$, $n_0 = N + 1$, $J \ge 1$, $T = 2^J$, use

$$B_n = 2C_a Q(\delta n_0^2) \frac{E(\delta n_0^2/2)}{1 - E(\delta n_0)},$$

$$B_t = \sum_{n=1}^{N} 2n^a T^3 E(\delta n^2 T) Q(\delta n^2 T).$$

Here $\delta n_0 \ge 6/5$ ensures a positive denominator; $E(x)$ equals one for $0 \le x < 1$, so this safeguard is essential. On the four panels of each $[2^j, 2^{j+1}]$, write midpoint m, halfwidth h, $\rho = m/2$, $v = h/\rho \le 2/9$, and $B(m) = \max\{(m + \rho)^2, (m - \rho)^{-2}, 1\}$. Use the Cauchy remainder

$$B_Q = \sum_{\text{panels}} \sum_{n=1}^{N} \frac{2n^a \, 2h \, 3B(m) E(\delta n^2 m/2) v^{2K+2}}{(2K + 3)(1 - v^2)}.$$

Choose in turn the first $N \ge 1$, first $J \ge 1$, and first $K \ge 0$ for which their respective budgets are at most $\eta/8$. Gaussian decay, then exponential time decay, then $v \le 2/9$ prove termination. These are the accepted tail and Cauchy arguments with each true exponential separately replaced by its upper bound E.

There are $P = 4JN$ potential pairs. A nonzero-character pair may be omitted only if its entire two-power contribution has bound

$$b = 4hn^a (m + h)^2 E(\delta n^2 (m - h)) \le \eta/(8P).$$

Add every omitted bound to B_{discard}, giving $B_{\text{discard}} \le \eta/8$. Evaluate all other integrated Taylor polynomials by the stated exact coefficient recurrence using outward rectangles. Refine both interval precision and all primitive series remainders until the final rectangle's sum of coordinate halfwidths satisfies $R_{\text{arith}} \le \eta/4$. For this fixed finite expression all input and primitive intervals shrink; continuity and outward arithmetic imply $R_{\text{arith}} \to 0$. Precision doubling therefore terminates, and the sum of the five raw components is at most $3\eta/4$. Choose an integer output scale B such that $5\, 2^{-B} \le \eta/8$ and round each component upward by $\text{up}_B(x) = 2^{-B} \lceil 2^B x \rceil$. The reported radius is their literal exact sum. With $\mathbf{b} = (R_{\text{arith}}, B_n, B_t, B_Q, B_{\text{discard}})$, it obeys

$$|y - \Lambda_q(s)| \le \sum_{i=1}^{5} \text{up}_B(b_i) \le \frac{3\eta}{4} + 5\, 2^{-B} \le \frac{7\eta}{8} < \eta.$$

The output retains exact rational midpoints, rounded components, B, raw interval radius, parameters, input metadata and source hashes. The raw analytic bounds are exactly reconstructible; they need

not be serialized as potentially enormous fractions. The output precision varies with the request and introduces no fixed reporting floor. □

This fixed-conductor sample result uses no hypothesis on zeros. It does not remove a downstream fixed-node Fourier alias floor, supply an all-conductor theorem or a positive arithmetic determinant, or prove GRH.

25 An end-to-end adaptive analytic readout

The new driver changes the sample truncations, interval precision, circle size and degree; it does not reuse the frozen labels to assert a smaller error. Its scope is still the three stated primitive quadratic characters and the fixed compact $|s - 1/2| \le 7/10$.

Theorem 25.1 (Requested-error construction in the stated finite family)**.** *Given a positive rational ε and $q \in \{3, 4, 5\}$, the specified exact-arithmetic algorithm terminates in the mathematical unbounded-resource model and returns a normalized rational polynomial P with*

$$\sup_{|s-1/2|\le 7/10} |P(s) - \Lambda_q(s)| \le \varepsilon.$$

Its sample enclosures, rational optimizer updates and all error budgets are explicit. This is an approximation theorem, with the stated classical theta identity as an input; it supplies no zero-location or positive determinant conclusion and is not a Lean theorem or a practical runtime bound.

Proof. Put $u = 7/12$, $v = 7/24$. Choose $d \ge 0$ so $10v^{d+1}/(1 - v) \le \varepsilon/8$, and choose $m > d$ so

$$10\frac{2^{-m}}{1 - 2^{-m}}\frac{1}{1 - v} \le \varepsilon/8.$$

The driver rounds m upward to a multiple of eight, which only decreases this bound. Set the requested sample error $\eta = \varepsilon(1 - u)/8$. The adaptive sampler gives all m true circle nodes and labels with error at most η. Form their outward Fourier rectangles through degree d, and refine the nodes and finite evaluation arithmetic if their weighted midpoint radii exceed $\varepsilon/8$. This refinement terminates: for these fixed finite d, m, node uncertainty and every finite-expression arithmetic width tend to zero with precision. Merely lifting fixed boxes would not suffice, so the driver regenerates the nodes and samples on this retry.

Choose rational midpoint targets t_n and execute the previously specified quadratic-objective updates until $\sum_{n=0}^{d} |b_n^k - t_n|_1 u^n \le \varepsilon/4$. Their geometric contraction proves finite termination. The oracle, aliasing, Taylor and Fourier-midpoint terms are each at most $\varepsilon/8$, and the optimization term is at most $\varepsilon/4$. Hence the raw five-term sum is at most $3\varepsilon/4$. Choose a serialization precision B with $5\,2^{-B} \le \varepsilon/8$, round each term upward, then add the five reported rationals exactly. The printed bound is at most $7\varepsilon/8 \le \varepsilon$. The polynomial uses $((s - 1/2)/(6/5))^n$, and the preceding Fourier/Cauchy proof gives its asserted compact-uniform bound. □

The trace is losslessly encoded by the fixed target vector and a rational scale extracted from each *executed* coefficient state, together with its objective, weighted residual and full-state hash. Every coefficient can be recovered exactly; a separate replay expands the states and checks every update. The producer does not generate its trace from a closed form. The saved objective remains the rational Fourier-midpoint quadratic, with its perturbation from ideal-root least squares explicitly budgeted.

26 An arithmetic-family moment with every original row retained

Let $K = \mathbb{Q}(\sqrt{-3})$, $\mathcal{O} = \mathbb{Z}[\omega]$, and write N for the ideal or element norm. Fix a finite-order Hecke character ν and a finite set S of prime ideals containing those above 6 and those dividing its conductor. For $0 \neq u \in \mathcal{O}$, let κ_u be the character of $K(u^{1/6})/K$ defined using arithmetic Frobenius, and put $\psi_u = \mathrm{prim}(\nu\kappa_u)$. Here prim means the underlying primitive Hecke character: it is formed before introducing any divisor variable or evaluating conductor-dependent zero extensions. The rows below are all actual Eisenstein elements, including units and prime powers supported in S.

The external inputs are abelian Kummer reciprocity and character uniqueness, the primitive Hecke functional equation and its usual vertical growth, Stirling's formula, scalar Dedekind-zeta convexity, and Blomer–Goldmakher–Louvel's large sieve for their specified sixth-order family. We use the pinned arXiv:1112.1650v1, Theorem 1.3, Lemma 2.1 and Appendix II; its Section 5 supplies the all-ideal decomposition whose fixed-twist proof is redone here. No statement about all primitive characters of comparable conductor, Artin holomorphy, Lindelöf bounds or zero locations is assumed.

Theorem 26.1 (The original-family numerator moment)**.** *For every $\varepsilon > 0$, uniformly for $H \geq 1$ and $t \in \mathbb{R}$,*

$$\sum_{0<Nu\leq H} |L(1/2+it,\psi_u)|^2 \ll_{\nu,S,\varepsilon} H^{1+\varepsilon}(1+|t|)^{1+\varepsilon}. \tag{8}$$

For every fixed $0 < \delta < 1/2$, the same estimate holds with $1/2+it$ replaced by $1/2+\delta+it$, with the constant also depending on δ. If the central bound has constant C_0 at an exponent $0 < \varepsilon_0 \leq 1/4$, an admissible off-central constant is

$$200C_0\left(\frac{3/2+\delta}{1/2-\delta}\right)^2. \tag{9}$$

Proof. For the canonical BGL character $\vartheta_{\mathfrak{a}}$, Appendix II gives a radicand $\alpha_{\mathfrak{a}}$ with $(\alpha_{\mathfrak{a}})_S = \mathfrak{a}\mathfrak{g}^{-6}$. Since $\mathcal{O}_S$ is principal, comparison with any literal u whose outside-S ideal is $\mathfrak{a}$ gives $\kappa_u = \eta\vartheta_{\mathfrak{a}}$ for one of finitely many fixed S-unit Kummer twists η. This is an equality of primitive characters, established outside a finite set and extended by uniqueness. For each outside-S ideal $\mathfrak{a}$ and S-supported integral ideal $\mathfrak{b}$, at most six elements generate $\mathfrak{a}\mathfrak{b}$. The resulting S-supported multiplicity sum converges at exponent $1+\varepsilon$.

For each fixed $\xi = \nu\eta$, split the canonical parameter as $\mathfrak{a} = \mathfrak{a}_0\mathfrak{a}_1\mathfrak{a}_2^6$, where $\mathfrak{a}_0$ is squarefree, $\mathfrak{a}_1$ is squarefull and sixth-power-free, and $(\mathfrak{a}_0, \mathfrak{a}_1) = 1$. The primitive conductor is at most a fixed multiple of $N(\mathfrak{a}_0\mathfrak{a}_1)$. The common-gamma approximate functional equation, with $X = H/(N\mathfrak{a}_2)^6$ and $T = 1+|t|$, has common maximal length $B = X^{1/2}T(HT)^{\eta}$ for any sufficiently small fixed $\eta > 0$. Mellin inversion removes its row-dependent conductor weight. Split a column as $\mathfrak{b}_S\mathfrak{b}_1\mathfrak{b}_2$, with $\mathfrak{b}_S$ supported in S, $\mathfrak{b}_1$ squarefree and $\mathfrak{b}_2$ squarefull, the last two coprime and prime to S. The original primitive values at $\mathfrak{b}_S$ remain bounded row multipliers. The S-supported and squarefull column weights are summable at $1/2+\eta$. Multiplying squarefree column coefficients by the fixed ξ preserves their squared norm, whatever its finite order. BGL's large sieve therefore bounds the energy for fixed $\mathfrak{a}_1, \mathfrak{a}_2$ by

$$(HT)^{O(\eta)}\{M + B + (MB)^{2/3}\}, \qquad M = X/N\mathfrak{a}_1.$$

Squarefull counting and summation give $X + B\sqrt{X} + (XB)^{2/3} \ll (HT)^{O(\eta)}XT$. The $\mathfrak{a}_2$ sum converges at exponent 6. Principal rows are separated before the pole-free functional equation and bounded by scalar zeta convexity. This proves (8).

For the off-central statement divide each $L(s,\psi)$ by $(s+1)^{5/4}$ and, on principal rows, also multiply by $(s-1)/(s+1)$. These are bounded analytic functions on $\mathrm{Re}\, s > 1/2$, continuous on

the boundary. Poisson representation and Jensen's inequality transfer the central squared moment. The convolution of $P_\delta(y) = \delta/[\pi(\delta^2 + y^2)]$ with $(1 + |t|)^{-3/2+\varepsilon_0}$ is at most eleven times that weight when $0 < \varepsilon_0 \le 1/4$. Restoring the denominator and the principal pole-removal factor gives (9). The extended edition gives the detailed derivation with all primitive local values. □

Separate fixed-unitary extension. The moment theorem also holds for any one fixed unitary Hecke character ν. For precision choose $\nu(\mathfrak{a}) = \nu_0(\mathfrak{a})(N\mathfrak{a})^{-i\tau}$, with angular parameter $m \in \mathbb{Z}$ and completed gamma factor

$$\gamma_\nu(s) = 2(2\pi)^{-(s+|m|/2+i\tau)}\Gamma(s + |m|/2 + i\tau).$$

All finite Kummer twists have this same infinity type. Use $T_\nu = 1 + |t + \tau| + |m|$, comparable to $1 + |t|$ with ν-dependent constants. Pure norm rows have $L(s, \psi) = \zeta_K(s + i\tau)$ and are separated before the pole-free functional equation; their pole at $1 - i\tau$ is removed by $(s - 1 + i\tau)/(s + 1 + i\tau)$. No uniformity in m, τ or the conductor of ν is asserted. This extension of the moment is separate from the following finite-order type-I residue formula.

26.1 The exact masked remainder and the remaining loss

Retain the original zero-extended symbol $\chi_d(u) = (u/d)_6$ on good squarefree ideals. Let μ be the ideal Möbius function and set

$$\beta_U(d) = \sum_{\substack{ab=d \\ Na, Nb \le U}} \mu(a)\mu(b), \quad T(u, d) = S \cup \operatorname{supp}(u) \cup \operatorname{supp}(d).$$

Let $\psi_{u,2} = \operatorname{prim}(\psi_u^2)$, formed independently of squaring any zero sentinel. With $\mathfrak{f}_u$ and $\mathfrak{f}_{u,2}$ the primitive conductors, the full-mask series is

$$F_{u,d}(s) = \frac{L(s, \psi_u)}{L(2s, \psi_{u,2})} C_{T(u,d)}(s), \tag{10}$$

$$C_T(s) = \frac{\prod_{p \in T,\ p \nmid \mathfrak{f}_u}(1 - \psi_u(p)(Np)^{-s})}{\prod_{p \in T,\ p \nmid \mathfrak{f}_{u,2}}(1 - \psi_{u,2}(p)(Np)^{-2s})}.$$

Initially it equals $\sum_{c \text{ squarefree},\ (c,T)=1} \psi_u(c)(Nc)^{-s}$ for $\operatorname{Re} s > 1$. Every prime of T remains deleted, including those forgotten by either primitive conductor.

For finite-order ν, $D \ge 1$, $U \ge 1$, $U^2 \le D$, and $W \in C_c^\infty((0, \infty))$ supported in $[1, 2]$, put

$$\widehat{W}(s) = \int_0^\infty W(x)x^{s-1}\,dx, \qquad \mathcal{M}_4(W) = \sum_{j=0}^{4} \sup_x |(x\partial_x)^j W(x)|.$$

On $\sigma = 1/2 + \delta$, the exact residue-subtracted type-I remainder is

$$E_u = \frac{D^\delta}{2\pi} \int_{\mathbb{R}} \widehat{W}(\sigma + it) D^{it} \sum_{\substack{d \text{ squarefree} \\ (d,S)=1}} \frac{\beta_U(d)\nu(d)\chi_d(u)}{(Nd)^{\sigma+it}} F_{u,d}(\sigma + it)\,dt. \tag{11}$$

This definition retains the actual β_U, all phases, d-masks, conductors and the independently primitive square.

Theorem 26.2 (An improved masked residual estimate)**.** *Under the preceding finite-order hypotheses, for every $\varepsilon > 0$,*

$$\sum_{0<Nu\le H} |E_u|^2 \ll_{\nu,S,\delta,\varepsilon} \mathcal{M}_4(W)^2 D^{2\delta} U^{2-4\delta} H(HD)^{\varepsilon}. \tag{12}$$

Proof. The same deleted set in both factors of (10) gives $|F_{u,d}(\sigma+it)| \ll (Nu\,Nd)^{\eta}|L(\sigma+it,\psi_u)|$: the reciprocal incomplete Euler product at $2\sigma > 1$ is bounded by $\zeta_K(2\sigma)$, and its remaining finite numerator is bounded by a small power of its radical. Apply Minkowski to (11) in the exact row space and use Theorem 26.1. The actual coefficient sum is at most

$$\sum_d \frac{|\beta_U(d)|}{(Nd)^{\sigma-\eta}} \le \left(\sum_{Na\le U} (Na)^{-\sigma+\eta} \right)^2 \ll U^{1-2\delta+2\eta}.$$

Four Mellin integrations by parts bound the remaining height integral by a constant times $\mathcal{M}_4(W)$. Square the result and absorb the small losses using $U^2 \le D$. □

At $H = D^{3/4}$ and $U = D^a$ with fixed $0 < a \le 1/2$, the upper bound's excess over H is $2\delta + (2-4\delta)a$, up to small losses; as $\delta \downarrow 0$ it still leaves $2a$. This is a limitation of the upper bound, not an energy lower bound. The extracted principal residue, the original short term at smaller scales and the signed type-II minus type-I covariance are not controlled by (12). The next arithmetic task is to estimate the complete d-weighted moment before its absolute coefficient sum, or establish sufficient signed covariance. No positive arithmetic determinant, zero-location theorem or GRH proof follows. The analytic argument uses the external classical inputs above; finite diagnostics and selected existing Lean local identities do not formalize the Hecke functions, approximate functional equation, Poisson transfer or these moment bounds.

27 Recombining the actual weighted cutoff

Keep the finite-order Eisenstein family and original masks of Section 26. For $U \ge 1$ and a squarefree ideal m, put

$$A_U(m) = \sum_{d|m} \beta_U(d) = \sum_{\substack{ab|m \\ (a,b)=1,\ Na,Nb\le U}} \mu(a)\mu(b).$$

Every summation variable here is an integral ideal. The last formula uses squarefreeness of m; an unconstrained square of a Möbius polynomial would also include forbidden overlapping supports.

Theorem 27.1 (Exact recombination and an absolute-half-plane rate)**.** *Write $T_0 = S \cup \mathrm{supp}(u)$ and $J_{u,U}(s) = \sum_{d \text{ squarefree},\ (d,S)=1} \beta_U(d)\nu(d)\chi_d(u)(Nd)^{-s}F_{u,d}(s)$. For $\mathrm{Re}\, s > 1$,*

$$J_{u,U}(s) = \sum_{m \text{ squarefree},\ (m,T_0)=1} A_U(m)\psi_u(m)(Nm)^{-s},$$
$$|A_U(m)| \le 3^{\omega(m)}, \qquad A_U(m) = \mu(m) \quad (Nm \le U).$$

For every $\kappa > 0$ and $0 < \epsilon < \kappa$, uniformly for $\mathrm{Re}\, s \ge 1+\kappa$, all original rows and all $U \ge 1$,

$$\left| J_{u,U}(s) - \frac{1}{L^{T_0}(s,\psi_u)} \right| \le U^{-\kappa+\epsilon}\{\zeta_K(1+\epsilon)^3 + \zeta_K(1+\epsilon)\}.$$

Here L^{T_0} removes every prime of T_0, including conductor-drop primes; the bound is in the absolute-convergence half-plane only.

Proof. On a surviving term $m = dc$, the two squarefree ideals c, d are coprime, and both avoid T_0. The original product $\nu(d)\chi_d(u)\psi_u(c)$ is then $\psi_u(m)$. Grouping by m gives A_U. Each prime of m has three possible positions in $(a, b, m/(ab))$, so the absolute sum is at most $3^{\omega(m)}$. When $Nm \le U$ all divisors fit both cutoffs; the primewise sum is $1 - 1 - 1 = -1$, giving $\mu(m)$.

The same bound justifies absolute convergence and regrouping for $\operatorname{Re} s > 1$, because on squarefree ideals $3^{\omega(m)} = \tau_3(m)$ and $\sum_m \tau_3(m)(Nm)^{-v} = \zeta_K(v)^3$ for $v > 1$, where this latter sum runs over all nonzero integral ideals. Subtract the absolutely convergent Möbius series for $1/L^{T_0}$. Only $Nm > U$ remain. Rankin's inequality with exponent $\eta = \kappa - \epsilon$ bounds their absolute sum by

$$U^{-\eta} \sum_m \{\tau_3(m) + 1\}(Nm)^{-1-\epsilon},$$

which proves the stated constant independently of deleted primes, row multiplicities or character phases. □

For W supported in $[1, 2]$, the same exact regrouping gives the original normalized type-I polynomial

$$I_u = D^{-1/2} \sum_{m \text{ squarefree},\ (m,S)=1} A_U(m)\nu(m)\chi_m(u)W(Nm/D).$$

This formula retains all original zero sentinels. It permits a direct family large sieve before any absolute d sum; its quantitative moment is developed separately below. The displayed absolute-half-plane rate alone is not that critical-strip moment.

28 A cutoff-uniform envelope for the original masked remainder

Fix $K = \mathbb{Q}(\omega)$, $\mathcal{O} = \mathbb{Z}[\omega]$, a finite-order Hecke character ν, and a finite set S containing the primes above 6 and the conductor of ν. All unspecified ideal sums below are over nonzero integral ideals. Let $\chi_m(u) = (u/m)_6$ be the original symbol, zero when $(m, u) > 1$, and put

$$\beta_U(d) = \sum_{\substack{ab=d \\ Na, Nb \le U}} \mu(a)\mu(b), \qquad A_U(m) = \sum_{d|m} \beta_U(d).$$

The columns d, c, m are squarefree and prime to S where indicated. Retain the primitive characters $\psi_u = \operatorname{prim}(\nu\kappa_u)$, $\psi_{u,2} = \operatorname{prim}(\psi_u^2)$, and the full deleted set $T(u, d) = S \cup \operatorname{supp}(u) \cup \operatorname{supp}(d)$. For $H, D, U \ge 1$, $U^2 \le D$, and $W \in C_c^\infty((0, \infty))$ supported in $[1, 2]$, the original normalized type-I piece is

$$I_u = D^{-1/2} \sum_{\substack{m \text{ squarefree} \\ (m,S)=1}} A_U(m)\nu(m)\chi_m(u)W(Nm/D). \tag{13}$$

Its exact principal residue is

$$P_u = \mathbf{1}_{\psi_u=1}\sqrt{D}\,\widehat{W}(1) \sum_{\substack{d \text{ squarefree} \\ (d,S)=1}} \frac{\beta_U(d)\nu(d)\chi_d(u)}{Nd}\rho_{T(u,d)},$$

$$\rho_T = \frac{\operatorname{res}_{s=1}\zeta_K(s)}{\zeta_K(2)} \prod_{p \in T}(1 + (Np)^{-1})^{-1}. \tag{14}$$

Define $E_u = I_u - P_u$; this is the same residue-subtracted remainder as the fully masked Mellin contour, with no cutoff limit or profile supremum.

Theorem 28.1 (Uniformity in the actual truncated cutoff)**.** *For every $\varepsilon > 0$,*

$$\sum_{0<Nu\le H} |E_u|^2 \ll_{\nu,S,\varepsilon} \|W\|_\infty^2 (HD)^\varepsilon \{H + DH^{1/3} + (HD)^{2/3}\}. \tag{15}$$

The sum is over every actual nonzero Eisenstein element, including units, S-supported powers and sixth-power rows. The constant is independent of U under $U^2 \le D$. This is an application of the classical sextic large sieve with exact original coefficients, not a zero-location theorem.

Proof. On an absolutely convergent line, expand the restored squarefree series $F_{u,d}$. Its condition $(c, T(u,d)) = 1$ includes $(c,d) = 1$, and the original symbols multiply. Grouping $m = cd$ gives (13) by Mellin inversion. Each prime of m may belong to the first truncated Möbius divisor, to the second, or to neither. Consequently $|A_U(m)| \le 3^{\omega(m)} \ll_\eta (Nm)^\eta$, uniformly in U. The length-D coefficient sequence in (13) therefore has squared norm $\ll_\eta \|W\|_\infty^2 D^\eta$.

The all-ideal row norm in Section 3 of Blomer–Goldmakher–Louvel, arXiv:1112.1650v1 gives

$$B_2^1(X, D) \ll_\eta (XD)^\eta \{X + DX^{1/3} + (XD)^{2/3}\}.$$

For completeness, split $a = a_1 a_2^2 \cdots a_5^5 a_6^6$ dyadically; $a_1, \dots, a_5$ are squarefree and pairwise coprime. If their dyadic sizes are A_j, put $R = \prod_{j=1}^6 A_j$ and $M = \max(A_1, A_2)$. Pivot on the first or second variable, using the squarefree powered sieve proved simultaneously in that paper's Section 3.2. The other variables are bounded column multipliers. The block energy is at most

$$(XD)^\eta \{R + DR/M + D^{2/3} R/M^{1/3}\} \sum_m |\lambda_m|^2.$$

The inequalities $R \le X$, $R^3 \le XM^3$ and $R^3 \le X^2 M$ follow from $\prod_j A_j^j \le X$ and the pivot choice. Summing the logarithmically many blocks proves the displayed bound.

Finite S-unit and ray-class partitions convert the original literal symbol into the canonical column value $\vartheta_m(a)$ with fixed modulus-one coefficient multipliers. On noncoprime pairs both values are zero: squarefree m has outside-S conductor exactly m. This orientation preserves sixth-power zeros. At most six elements generate each outside-S ideal times an S-supported ideal; the S-supported sums converge at powers 1, 1/3 and 2/3. Thus (15) holds first with I_u in place of E_u.

A principal ψ_u requires every outside-S valuation of u to be divisible by 6. There are $O_{\nu,S}(H^{1/6})$ such literal rows. Equation (14), $|\rho_T| \ll_K 1$, and the actual coefficient sum

$$\sum_d \frac{|\beta_U(d)|}{Nd} \le \left(\sum_{Na \le U} \frac{1}{Na} \right)^2 \ll \log^2(2U)$$

give $\sum |P_u|^2 \ll DH^{1/6} \|W\|_\infty^2 \log^4(2U)$. This is absorbed by the right side of (15), using $U^2 \le D$ and a smaller internal exponent. Finally $|E_u|^2 \le 2|I_u|^2 + 2|P_u|^2$. The extended proof specifies the canonical-family transfer and the powered-sieve input precisely. □

Combining this with the previously proved fixed-twist AFM transfer yields

$$\sum_{0<Nu\le H} |E_u|^2 \ll_{\nu,S,\delta,\varepsilon} \mathcal{M}_4(W)^2 (HD)^\varepsilon \min\{D^{2\delta} U^{2-4\delta} H,\ H + DH^{1/3} + (HD)^{2/3}\}. \tag{16}$$

At $H = D^{3/4}$ and $U = D^a$, $0 \le a \le 1/2$, the excess over H is at most $\min\{2\delta + (2-4\delta)a, 1/2\}$, up to arbitrarily small losses. The new envelope improves the old one for $a > 1/4$ as $\delta \downarrow 0$, but still leaves an exponent loss of 1/2. It does not settle the original signed short-row bound, type-II covariance, positive arithmetic determinant comparison or GRH. The principal-residue bound used here is sufficient for this envelope, not the stronger target needed to reassemble the full signed argument.

29 High cutoffs, actual bilinear factors and a signed covariance test

Retain the original Eisenstein family, fixed finite-order ν, fixed S, common $W \in C_c^\infty((0,\infty))$ supported in $[1,2]$, and every raw shared-prime zero. Put

$$C_U(m) = \sum_{\substack{abc=m \\ Na>U,\, Nb>U}} \mu(a)\mu(b), \qquad r_U(m) = \#\{(a,b) : ab = m,\ Na, Nb > U\}.$$

All factors are ideals, squarefree and pairwise coprime; on the active annulus $A_U = C_U - \mu$. A coefficient identity is different from a row-energy bound.

Proposition 29.1 (The high-cutoff window)**.** *If $D > 1$ and $2D/7 \le U^2 \le D$, then for every surviving squarefree column $D \le Nm \le 2D$,*

$$C_U(m) = \mu(m) r_U(m), \qquad A_U(m) = \mu(m)(r_U(m) - 1), \qquad |A_U(m)| \ge 1.$$

For each fixed nonzero W, the normalized squared coefficient norm satisfies

$$D^{-1} \sum_{\substack{m \text{ sf} \\ (m,S)=1}} |A_U(m) W(Nm/D)|^2 \gg_{W,S} 1 \tag{17}$$

for large D, uniformly in this window. This is not a lower bound for row energy.

Proof. The least norm of a nonunit ideal outside S is at least 7. A long-long term with $c \ne 1$ would have $Nm > 7U^2 \ge 2D$. Thus $ab = m$, including both endpoints. The fixed-point-free swap $(a,b) \mapsto (b,a)$ makes r_U even, so A_U is an odd nonzero integer. The last claim follows by counting squarefree ideals on a fixed interval where $|W|$ is bounded below; the constant and error are proved in the extended proof. □

The exact remaining polynomial is genuinely bilinear, with $(a,b) = 1$, $U < Na, Nb < 2D/U$, $D \le Na\, Nb \le 2D$, and $x_{ab}(u) = x_a(u)x_b(u)$, including zeros. Both factors see the same row $u = \varepsilon\alpha\beta^2$. Cauchy followed by the accepted powered BGL bound gives only

$$Q_{B_X}(C_U, C_U) \ll_{\nu,S,\epsilon} (DX)^\epsilon \|W\|_\infty^2 \{\sqrt{D}\, X^2 + DX + D^{5/6} X^{5/3}\}. \tag{18}$$

At $X \asymp D^{1/4}$ the exponents are $1, 5/4, 5/4$; the earlier regrouped bound has $1/2, 5/4, 13/12$ and is stronger. No BT_σ saving is obtained. Multiplying two marginal operator norms does not supply a coupled fourth moment.

A finite test in the original family. There is no automatic nonnegative signed covariance, even when $C_U(m)\mu(m) \ge 0$. Take $\nu = 1$, S the primes above $2, 3$, $D = 400$, $U = 17$, and all 72 literal rows of B_7. Choose the common nonnegative smooth profile constructed in the extended proof, with values 10 at $439/400$, 1 at $475/400$, and zero at every other integer norm divided by 400. There are exactly four active squarefree ideal columns, of norms $439, 439, 475, 475$. In that order,

$$A = (1,1,1,1), \quad C = (0,0,2,2), \quad \mu = (-1,-1,1,1), \qquad \Re G_{B_7} = \begin{pmatrix} 72 & 0 & 0 & 18 \\ 0 & 72 & 18 & 0 \\ 0 & 18 & 72 & 0 \\ 18 & 0 & 0 & 72 \end{pmatrix}.$$

Exact raw-symbol enumeration, independently reproducible from the ideal and residue-field definitions, gives

$$Q(C,\mu)=-\frac{27}{25},\qquad Q(C,C)=\frac{36}{25},\quad Q(\mu,\mu)=\frac{864}{25},\quad Q(A,A)=\frac{954}{25}. \tag{19}$$

The real Gram matrix is positive definite, with eigenvalues $54, 90$; its positivity does not force a cross covariance to be positive. This fixed finite example refutes that shortcut, not the asymptotic BT_σ target or the original signed mean-square estimate.

The two-cutoff contour remains coupled. For $T_0 = S \cup \operatorname{supp}(u)$, $\psi = \psi_u$ and independently primitive $\psi_2 = \operatorname{prim}(\psi^2)$, put $q_p = \psi(p)(Np)^{-s}$, $\alpha_p = (Np)^{-z}$, $\beta_p = (Np)^{-w}$ outside T_0. The actual three-placement Euler factor is $1 + q_p - q_p\alpha_p - q_p\beta_p$. Its exact factorization is

$$\mathcal{F}(s,z,w)=\frac{L^{T_0}(s,\psi)}{L^{T_0}(s+z,\psi)L^{T_0}(s+w,\psi)}\prod_{p\notin T_0}\left(1-\frac{q_p^2(1-\alpha_p)(1-\beta_p)}{(1-q_p\alpha_p)(1-q_p\beta_p)}\right). \tag{20}$$

Exact double Perron inversion is valid on positive auxiliary lines $c, d > 1 - \Re s$ for $1/2 < \Re s < 1$, retaining the original principal pole exclusion. These safe lines reproduce the previous $U^{2-4\delta}$ energy loss at $\Re s = 1/2 + \delta$, with the same common-profile derivative and fixed-margin dependencies. The correction product alone does not justify moving those lines through reciprocal-L poles. In fact $\mathcal{F}(s,z,0) = 1/L^{T_0}(s+z,\psi)$ as a meromorphic identity. No zero-free region, reciprocal moment, analytic saving, new Lean formalization or GRH proof is supplied by this construction.

30 A quantitative covariance budget without a new exponent

Retain the original Eisenstein sextic symbols, fixed finite-order ν, fixed S, one possibly complex common $W \in C_c^\infty$ supported in $[1,2]$, all squarefree columns prime to S, and every raw shared-prime zero. In the accepted window $2D/7 \le U^2 \le D$, put $r_m = r_U(m)$, $\lambda_m = \mu(m)\nu(m)W(Nm/D)$ and $v_m = \overline{\lambda_m}$. Thus $C_U = \mu r$ and $A_U = \mu(r-1)$ on the active annulus. For a finite literal row block B, let

$$G_{mn}=\sum_{u\in B}x_m(u)\overline{x_n(u)},\qquad R=\operatorname{diag}(r_m),\qquad M_j=\sum_m r_m^j|W(Nm/D)|^2\quad(0\le j\le 2).$$

The primitive $\psi_u, \psi_{u,2}$ and full deleted sets retain their previous meanings; no conductor-drop value replaces a raw zero.

Proposition 30.1 (An actual-vector lower budget)**.** *If $0 \le G \le \Lambda_B I$, then*

$$\Re Q_B(C_U,\mu)\ge-\frac{\Lambda_B}{2D}\{\sqrt{M_0M_2}-M_1\}. \tag{21}$$

In this high window, uniformly in U,

$$\sum_{D\le Nm\le 2D}r_m\ll_K D,\qquad \sum_{D\le Nm\le 2D}r_m^2\ll_K D\log(2D). \tag{22}$$

Consequently

$$\Re Q_B(C_U,\mu)\ge -C_K\|W\|_\infty^2\Lambda_B\sqrt{\log(2D)}.$$

The vector inequality is sharp among positive contractions with the specified v, Rv; no such sharpness is asserted for the arithmetic Gram. The moment bound counts actual balanced ideal factorizations, rather than assuming independence of the two row factors.

On B_X, inserting the previously accepted powered-BGL norm gives only

$$\Re Q_{B_X}(C_U,\mu) \geq -C_{\nu,S,\epsilon}\|W\|_\infty^2 (DX)^\epsilon \{X^2 + DX + D^{2/3}X^{5/3}\}.$$

At $X \asymp D^{1/4}$ the exponents remain $1/2, 5/4, 13/12$. At $H = D^{3/4}$ this budget still has the old $D^{1/2}$ excess over H; it supplies no BT_σ saving. A negative-part budget is weaker than $2\Re Q(C_U,\mu) \geq Q(C_U,C_U) + Q(\mu,\mu) - R_\sigma$.

For $S_R = \sqrt{R}$, the exact identity

$$\tfrac{1}{2}(RG + GR) = S_R G S_R + \tfrac{1}{2}[S_R,[S_R,G]]$$

localizes the commutator to unequal factor-count levels, with entry $(\sqrt{r_m} - \sqrt{r_n})^2 G_{mn}$. If B retains all six literal units, the actual unit sum further gives $G_{mn} = 0$ unless $Nm \equiv Nn \pmod{36}$. These are exact deletions, not a power saving. Even zero different-level pairings leave the nonnegative within-level energies $\sum_k (k-1)^2 Q_B(\mu \mathbf{1}_{r=k}, \mu \mathbf{1}_{r=k})$. The unequal-product coupled fourth moment and its mixed Möbius term remain unestimated. No new analytic exponent, energy lower bound, Lean formalization or GRH conclusion follows.

31 The fourth-moment conductor and its full raw zero set

Retain the original Eisenstein sextic symbols $x_m(u) = (u/m)_6$, fixed finite-order ν, fixed S, and one possibly complex common smooth W supported in $[1,2]$. Put $\lambda_m = \mu(m)\nu(m)W(Nm/D)$ and $r_m = r_U(m)$. Columns remain squarefree and prime to S. In $2D/7 \leq U^2 \leq D$, the actual polynomial is

$$\mathcal{C}_U(u) = \sum_{\substack{(a,b)=1,\ Na,Nb>U \\ D \leq Na\,Nb \leq 2D}} \mu(a)\mu(b)\nu(ab) W(Nab/D) x_a(u) x_b(u).$$

Its squared row energy has four factor indices a, b, c, d, with $(a,b) = (c,d) = 1$. These internal masks are essential. At each prime,

$$e_p = \mathbf{1}_{p|a} + \mathbf{1}_{p|b} - \mathbf{1}_{p|c} - \mathbf{1}_{p|d} \quad \text{belongs to } \{0, \pm 1\}.$$

Thus, for the two squarefree products $m = ab, n = cd$, the primitive oscillatory conductor outside fixed S is $q = mn/(m,n)^2$. The full raw zero-preserving period is $L = mn/(m,n)$:

$$x_m(u)\overline{x_n(u)} = \mathbf{1}_{(u,(m,n))=1} \theta_{m/(m,n),\, n/(m,n)}(u).$$

No common-prime zero is restored by taking the primitive character.

A generic fourth moment $|\sum_m \lambda_m x_m(u)|^4$ has four individually squarefree columns $m_1, \ldots, m_4$ with no such internal coprimality. Its signed local exponent $e_p = \sum_{i=1}^2 \mathbf{1}_{p|m_i} - \sum_{i=3}^4 \mathbf{1}_{p|m_i}$ can also be ± 2, giving a nontrivial cubic component. For $L = \mathrm{rad}(m_1 m_2 m_3 m_4)$, the exact row-power interface is

$$Q_j = \prod_{\substack{p|L \\ j e_p \not\equiv 0 \ (6)}} p, \qquad Z_j = L/Q_j, \qquad \prod_{i=1}^2 x_{m_i}(v^j) \overline{\prod_{i=3}^4 x_{m_i}(v^j)} = \theta_{j,e}(v) \mathbf{1}_{(v,Z_j)=1}.$$

Here $\theta_{j,e}$ is primitive on Q_j, with its own zero extension. At $j = 3$ an exponent ± 2 becomes primitive-principal but its raw zero remains. For the original two-squarefree-product expansion,

$Q_j = q$ for all $1 \le j \le 5$, including the quadratic $j = 3$ direction. For the accepted row blocks $Nr_j \asymp A_j$, put $R_0 = \prod_j A_j$, $P = \max(A_1, \dots, A_5)$ and retain $\prod_j A_j^j \le H$. The full A_3 saturation face and all row-factor masks remain.

The small-conductor pair count recovers an already controlled high-gcd sector; it is not a new analytic advance. On the remaining original sector $Nq > P^2$, favorable smooth completion costs $\min(A_j, \sqrt{Nq}) = A_j$ for $1 \le j \le 5$. The literal sharp cutoff is not replaced by this smooth trial. The signed large-conductor fourth-moment and mixed Möbius sums remain open; no improved full exponent, BT_σ, new Lean theorem, zero-location result or GRH conclusion follows.

32 A quadratic pivot on the cube factor

Retain the original zero-extended sextic symbols $\chi_m(u) = (u/m)_6$ over $K = \mathbb{Q}(\omega)$, the fixed S containing all primes above 6 and comparison moduli, and the squarefree good columns $D \le Nm \le 2D$. Write an outside-S row ideal uniquely as $r_1 r_2^2 r_3^3 r_4^4 r_5^5 r_6^6$: only $r_1, \dots, r_5$ are pairwise-coprime squarefree ideals; r_6 is arbitrary and may overlap them. Use primary generators. Fix one literal unit and one S-supported row factor, retaining their original values. Let $A_j \le Nr_j < 2A_j$, $A_j \ge 1$, $R_0 = \prod_j A_j$ and $\prod_j A_j^j \le H$; the actual capacity restriction on rows is retained. Coefficients $b_m \in \mathbb{C}$ are arbitrary and fixed independently of every variable row.

Proposition 32.1 (The cube-factor block norm)**.** *For every $\varepsilon > 0$, the raw Gram on this actual block obeys*

$$\sum_{u \text{ in the block}} \left| \sum_m b_m \chi_m(u) \right|^2 \ll_{\nu,S,\varepsilon} (HD)^\varepsilon \left\{ R_0 + \frac{DR_0}{A_3} \right\} \sum_m |b_m|^2.$$

The constant is uniform in the fixed unit/S-factor, frozen good factors, H, D and the coefficient sequence. There is no rational-norm-squarefree restriction. Consequently, with $M = \max(A_1, A_2)$, the old powered-BGL bound and this bound give

$$\Lambda_A \ll_{\nu,S,\varepsilon} (HD)^\varepsilon \left\{ R_0 + \min\left(\frac{DR_0}{M} + \frac{D^{2/3} R_0}{M^{1/3}}, \frac{DR_0}{A_3} \right) \right\}. \tag{23}$$

The external analytic input is Goldmakher–Louvel, arXiv:1112.1642v2, Corollary 1.2 with $n = 6$, already an explicit classical cube-power sieve. The refinement is this application, not a new large-sieve theorem. For the original normalized coefficients $b_m = D^{-1/2} A_U(m) \nu(m) W(Nm/D)$, the accepted divisor bound gives $\sum_m |b_m|^2 \ll_\eta \|W\|_\infty^2 D^\eta$, uniformly in $U \ge 1$ with $U^2 \le D$ (and the common W may be complex). Choosing smaller internal exponents yields the corresponding uniform block energy envelope. General cutoff or factor-count weights remain inside the arbitrary coefficient norm; that norm is not silently replaced by one. Summing S-supported row factors uses the reduced capacity $H/N\mathfrak{b}$ and the established convergent geometric sums.

On the complete old saturated face $A_1 = A_2 = A$, $A_3 = B$, $A_4 = A_5 = A_6 = 1$, $AB = H^{1/3}$, the new bound is $A^2 B + DA^2$ in place of the old middle monomial DAB. For $H = D^{3/4}$ put $A = D^a$, $B = D^{1/4-a}$, $0 \le a \le 1/4$. When $a < 1/8$, its leading exponent is $1 + 2a < 5/4$; the gain requires B/A to grow by a power for a power improvement. For $a \ge 1/8$, the combined bound still has leading exponent $5/4$. Thus the A_3-dominant part improves, while no new global exponent, BT_σ estimate, signed covariance theorem, Lean theorem or GRH conclusion follows.

Inspecting the surviving sector. The quadratic-pivot inspector plots the complete face, the five monomial powers and the remaining target gap. These are analytic bound exponents, not

measured operator norms or trained-network predictions. The refined norm can be inserted in the earlier covariance budget; it does not remove its within-level energies.

A necessary near-face condition. Put $\mathcal{M}_A = DR_0/\max(M, A_3)$. The combined envelope is at most $(HD)^{\varepsilon}\{R_0 + \mathcal{M}_A + D^{2/3}R_0/M^{1/3}\}$. If $\mathcal{M}_A \geq DH^{1/3}D^{-\eta}$, then the earlier near-face slack is at most $3\eta \log D$ and, additionally, $A_3 \leq MD^{\eta}$. Thus a uniform remaining leading-term estimate may focus on this smaller neighborhood; it must still cover the entire $B \leq A$ part of the face. This is a necessary condition on explicit upper-bound monomials, not a lower bound for actual arithmetic energy.

33 Joint completion before conductor absolute values

In the original C–C expansion write $m = hr, n = ht$ with h, r, t pairwise coprime squarefree, $q = rt \neq 1$, $L = hq$, $Q = Nq$, $\ell = NL$. Set $\theta = x_r\overline{x_t}$ and $\xi_q = x_q^3$. The original internal masks give $\theta^3 = \xi_q$ and primitive θ^j on q for $1 \leq j \leq 5$, while $\mu(hr)\mu(ht) = \mu(q)$. Fix generators α of h, β of q, choosing $\alpha = 1$ for $h = 1$. With $\delta = 1 + 2\omega$, put $e_\beta(z) = \exp(2\pi i \operatorname{Tr}(z/(\delta\beta)))$, $\tau_\beta(\eta) = \sum_{z \bmod q} \eta(z)e_\beta(z)$, and $c_h(k) = \prod_{p|h}(Np\mathbf{1}_{k\in p} - 1)$, $c_1 = 1$. For $f_j(v) = \theta(v)^j\mathbf{1}_{(v,h)=1}$, $1 \leq j \leq 5$, $f_6(v) = \mathbf{1}_{(v,L)=1}$ and $G_j(k) = \sum_{v \bmod L} f_j(v)e_{\alpha\beta}(kv)$, joint completion gives

$$\prod_{j=1}^{6} G_j(k_j) = Q^2\xi_q(-\alpha)\tau_\beta(\xi_q)\overline{\theta(\prod_{j=1}^{5} k_j^j)} \prod_{j=1}^{5} c_h(k_j)c_L(k_6). \tag{24}$$

This holds when the first five frequencies are q-units; otherwise the left side is zero. The mixed Gauss phase is orientation-independent, with every common-prime zero retained. The exact row Fourier weight still contains all squarefree, ray, inter-factor, capacity and profile restrictions, including the full A_3 face. Its weighted joint conductor/frequency estimate remains open. This identity and its finite Parseval budget give no improved global exponent or GRH conclusion.

34 A radical-conductor refinement of the raw envelope

Retain the original literal symbols $\chi_m(u) = (u/m)_6$, extended by zero, the fixed good set S, and the squarefree ideal columns $D \leq Nm \leq 2D$. On an actual dyadic row block write

$$u = u_0r_1r_2^2r_3^3r_4^4r_5^5r_6^6, \qquad A_j \leq Nr_j < 2A_j, \qquad \prod_{j=1}^{6} A_j^j \leq H.$$

Only $r_1, \ldots, r_5$ are pairwise-coprime squarefree ideals; r_6 is arbitrary and may overlap them. Fix primary generators, a literal unit and an S-supported factor in u_0. Put $R = \prod_j A_j$, $M = \max(A_1, A_2)$ and $Q = R/A_6$. For arbitrary complex coefficients b_m, independent of every variable row, define

$$\mathcal{E}_A(b) = \sum_{u \text{ in the actual block}} \left|\sum_m b_m\chi_m(u)\right|^2, \qquad \Lambda_A = \sup_{b\neq 0} \frac{\mathcal{E}_A(b)}{\sum_m |b_m|^2}.$$

Proposition 34.1 (An ordinary conductor comparison)**.** *For every $\varepsilon > 0$,*

$$\Lambda_A \ll_{S,\varepsilon} (HD)^{\varepsilon}A_6\{D + Q^2\}. \tag{25}$$

Consequently the original residue-subtracted remainder satisfies, uniformly for $H, D, U \geq 1$ and $U^2 \leq D$,

$$\sum_{0<Nu\leq H} |E_u|^2 \ll_{\nu,S,\varepsilon} \|W\|_\infty^2 (HD)^\varepsilon \{H + DH^{1/6} + (DH)^{2/3}\}. \tag{26}$$

Here ν, S are fixed and $W \in C_c^\infty((0,\infty))$ is one common, possibly complex profile supported in $[1,2]$; there is no profile supremum.

The input to (25) is the classical ordinary sieve for *all* primitive characters of conductor norm at most Q, with constant $D + Q^2$, not a higher-order-family theorem for arbitrary conductor labels. The following section gives its standard Gauss-sum and two-dimensional lattice proof, including prime-power conductors at S. For fixed r_6, the first five factors have primitive outside-S conductor $q = r_1 \cdots r_5$. The literal primitive character agrees with the raw symbol even at collisions with q; the sixth-power factor instead remains the coefficient mask $\mathbf{1}_{(m,r_6)=1}$. At most $5^{\omega(q)} \ll_\eta (Nq)^\eta$ tuples give the same primitive character. Applying the ordinary sieve and then counting r_6 proves (25). No rational-norm-squarefree restriction is imposed.

To obtain (26), combine this with the accepted powered-BGL block bound from Section 28. With $S_A = R/M$ and $T_A = D^{2/3}R/M^{1/3}$, capacity gives $R \leq H$, $A_6 \leq H^{1/6}$, $RS_A \leq H$ and $T_A \leq (DH)^{2/3}$. Hence

$$\begin{aligned}\min(R + DS_A + T_A, DA_6 + R^2/A_6) &\leq R + T_A + DA_6 + \min(DS_A, H^2/S_A^2) \\ &\leq H + DH^{1/6} + 2(DH)^{2/3}.\end{aligned}$$

Dyadic and fixed-S summation, the uniform actual coefficient norm, and the principal-residue estimate give the assertion for $E_u = I_u - P_u$.

At $H = D^{3/4}$ the new envelope has exponent $7/6$, improving the old $5/4$ by $1/12$; its excess above the target H is still $D^{5/12}$. On the *entire* old face $A_1 = A_2 = A$, $A_3 = B$, $A_4 = A_5 = A_6 = 1$, $AB = H^{1/3}$, one has $Q = A^2B = AH^{1/3} \leq H^{2/3}$, so (25) costs $D^{1+\varepsilon}$ there. This includes $B \leq A$. The new combined upper bounds reach $7/6$ at $(\log_D A_1, \log_D A_2) = (5/12, 1/6)$ with other coordinates zero; the pure point $\log_D A_1 = 3/4$ also retains the old BGL cross term. These are upper-envelope witnesses, not arithmetic energy lower bounds or counterexamples to signed cancellation.

The simpler proposed replacement of the radicand ab^2 by its radical ab inside BGL remains unjustified: after the existing fixed reciprocity phases, the mixed reciprocity quotient is $\vartheta_b(a')/\vartheta_a(b')$, which varies outside S. The present application does not require that replacement. No target signed moment, type-II covariance estimate, positive arithmetic determinant comparison, zero-location result, new Lean theorem or GRH conclusion follows from (26).

35 Learning the actual signed arithmetic correlations

The unsigned refinement in Section 34 leaves a signed problem. Here is a separate, executed finite learning objective on the original correlations, with a certificate for every coordinate and observable. It is not a uniform estimate for an expanding family. For ordered distinct squarefree columns $m = hr, n = ht$, put $q = rt$ and

$$\Gamma_i = \sum_{u\in\mathcal{R}} \chi_r(u)\overline{\chi_t(u)}\mathbf{1}_{(u,h)=1}, \qquad i = (h,q,r).$$

The symbols vanish at every collision with q; the common-prime mask is retained separately. With principal ν, actual complementary divisor counts $r_U(m)$, and one common complex profile

$W_m = W(Nm/D)$, define

$$d_i^{CC} = r_U(m)r_U(n)W_m\overline{W_n}, \qquad d_i^{C\mu} = r_U(m)W_m\overline{W_n}, \qquad \lambda_i = 1 + |d_i^{CC}|^2 + |d_i^{C\mu}|^2.$$

The exact diagonal Δ_X and these offdiagonal coordinates give

$$E_X = \Delta_X + D^{-1}\sum_i \mu(q_i)d_i^X\Gamma_i, \qquad X \in \{CC, C\mu\}. \tag{27}$$

Thus $E_{CC} = \sum_{u\in\mathcal{R}} |C_u|^2$ and $E_{C\mu} = \sum_{u\in\mathcal{R}} C_u\overline{M_u}$ for the actual normalized row sums. Neither the Möbius sign nor a complex conjugate is discarded.

Proposition 35.1 (A finite supervised certificate)**.** *Let g_i be complex readout coordinates and $L(g) = \sum_i \lambda_i|g_i - \Gamma_i|^2$. For $K_X = \sum_i |d_i^X|^2/\lambda_i$ and $\lambda_{\min} = \min_i \lambda_i$, the substituted readouts satisfy*

$$|\widehat{E}_X - E_X|^2 \le D^{-2}K_X L(g),$$
$$\max_i |g_i - \Gamma_i|^2 \le L(g)/\lambda_{\min}.$$

The executed metric-gradient update $g^{(t+1)} = (g^{(t)} + \Gamma)/2$ gives $L(g^{(t)}) = 4^{-t}L(g^{(0)})$. Therefore every positive rational requested tolerance on these three squared bounds is attained after finitely many updates on this fixed exact instance, with unrestricted resources.

The implementation's resource guard refuses requests beyond its update budget; termination in this proposition is not a practical runtime guarantee. The labels Γ_i are computed exactly before training, so no speedup over exact label evaluation or generalization to unseen conductors is claimed.

The supplied instance uses $D = 6000, U = 60, H = 625$, S above $2, 3$, principal ν, and all 15 good squarefree ideal columns at activated norms $7507, 7519, 8281, 10507$ (respectively $2, 4, 1, 8$ ideals). One explicitly specified common smooth four-bump profile is zero at every other integer norm. Both the 2262 literal rows and the separately labelled 2148 guaranteed nonprincipal subset are evaluated; the latter is not asserted to be the whole nonprincipal family. There are 95 offdiagonal (h, q) blocks, 65 with $h = 1$, and 210 ordered learned coordinates. Every row case retains 4530 raw column zeros. Real and complex profiles give four requested-10^{-12} certificates, using $53, 55, 53, 55$ metric updates respectively: 216 executed updates, with every full rational state saved. A separate 200 executed ordinary coordinate updates are budget-limited and carry no requested-error claim. The standalone replay is linked from the book and project page. No analytic saving, positive determinant, zero-location assertion, upstream quantum training, new Lean declaration or GRH conclusion follows.

36 Why the common profile must stay fixed

The unsigned envelope of Section 34 and the unresolved signed estimate have different profile quantifiers. We give an obstruction to an H-sized bound uniform through the sup norm alone. It uses the *actual* coefficients and nonprincipal rows, rather than replacing the columns by arbitrary weights. It does not contradict an estimate for one fixed smooth profile or one with adequate derivative dependence.

Fix $K = \mathbb{Q}(\omega)$, $\omega^2 + \omega + 1 = 0$, principal $\nu = 1$, and a finite good set S containing the primes above $2, 3$. Write $\zeta = 1 + \omega$, $\delta = 1 + 2\omega$, so ζ has order six and $\delta^2 = -3$. Retain the literal raw symbols and normalization

$$I_u(W) = D^{-1/2} \sum_{m \text{ squarefree } (m,S)=1} A_U(m)\chi_m(u)W(Nm/D), \qquad E_u = I_u - P_u.$$

Here and below squarefree means *ideal* squarefree. For $U^2 = D$ the accepted high-window identity gives $A_U(m) = \mu(m)(r_U(m) - 1)$ on $D < Nm < 2D$, with $r_U(m)$ even and $|A_U(m)| \geq 1$. The multiset of prime-ideal norms of a squarefree ideal is determined by its rational norm. Thus $A_U(m) = A_N$ depends only on $N = Nm$, including divisor-norm multiplicities. Moreover

$$\eta(m) := \chi_m(\zeta) = \zeta^{(Nm-1)/6}, \qquad (m, S) = 1. \tag{28}$$

The primitive character η has conductor $(2)^2(\delta)^3$, of norm 432; its powers $1, 5$ have conductor norm 432, powers $2, 4$ norm 27, and power 3 norm 16. All five powers are nonprincipal. These row conductors do not remove the raw zero at any prime dividing a sixth factor.

Put $\phi(t) = \exp(1 - 1/(1 - t^2))$ for $|t| < 1$ and zero otherwise. For integer D set $w_N = \operatorname{sgn}(A_N)\overline{\zeta^{(N-1)/6}}$ when a good squarefree ideal of norm N exists and $D < N < 2D$, and zero otherwise. Define one common, complex profile

$$W_D(y) = \sum_{D<N<2D} w_N \phi(4(Dy - N)). \tag{29}$$

Its disjoint supports lie strictly inside $(1, 2)$, its sup norm is one for large D, and $\|W_D^{(j)}\|_\infty \ll_j D^j$. In particular the endpoints have value zero. This is a profile varying with scale, independent of the row, not a separate choice for each ideal or each row.

Proposition 36.1 (A moving-profile lower bound)**.** *For every fixed $0 < \theta < 1$, take $H = D^\theta$, $U = \sqrt{D}$, and $X = H^{1/6}$. As integer $D \to \infty$,*

$$\sum_{0<Nu\leq H} |E_u(W_D)|^2 \geq (c_S + o(1))DH^{1/6}, \qquad c_S > 0. \tag{30}$$

This lower bound already comes from literal rows $u_v = \zeta\beta_v^6$, where v is an arbitrary good ideal of norm at most X and β_v is a generator. These rows have nonprincipal primitive character η, so $P_{u_v} = 0$. Their exact values are

$$E_{u_v}(W_D) = D^{-1/2} \sum_{\substack{m \text{ squarefree, } (m,Sv)=1 \\ D<Nm<2D}} |A_U(m)|. \tag{31}$$

Every collision with v remains deleted, even when the primitive conductor drops to 432. No squarefree condition is imposed on v.

At $H = D^{3/4}$ the lower exponent is $9/8$, whereas the current upper exponent is $7/6$. Thus this does not establish sharpness of $7/6$ or a new upper saving. For $0 < \theta < 2/3$, the term $DH^{1/6}$ dominates the current unsigned envelope; it is necessary up to subpower losses in this *varying-profile* class, even on nonprincipal rows.

For $k \geq 0$ put $M_k(W) = \sum_{j=0}^k \|(y\partial_y)^j W\|_\infty$. If, for every $\varepsilon > 0$, a bound uniform over all common profiles with constants independent of W, D were to assert

$$\sum_{0<Nu\leq H} |E_u(W)|^2 \ll_\varepsilon M_k(W)^\alpha (HD)^\varepsilon H, \qquad \alpha \geq 0,$$

then necessarily $k\alpha \geq 1 - 5\theta/6$. At $\theta = 3/4$ this is $k\alpha \geq 3/8$. For a single fixed ε the necessary condition is only $k\alpha + (1 + \theta)\varepsilon \geq 1 - 5\theta/6$. These are necessary cost comparisons, not sufficient estimates. A fixed-W result, a class with bounded seminorms, or the existing $M_4(W)^2$ dependence is not refuted.

The same rows expose a concrete centering error in the joint completion: removing the orientation mean of the actual correlation can delete a nonzero signed contribution on *nonprincipal* rows. The extended proof retains the literal unit phase, gives an exact four-orientation example, and explains why orientation projection alone supplies no saving. The standalone profile and coherent-mean inspector shows the complete finite norm groups, arbitrary sixth-factor masks and exact residual quantities. Its finite tables are diagnostics of this construction, not asymptotic estimates, training or GRH evidence.

37 A bound for the signed coprime slice and an exact character twist

The coprime-column contribution can be negative, so it needs its own bound. Fix the original ν, S, one common possibly complex smooth W supported in $[1,2]$, $D>1$, $U\geq 1$ and $U^2\leq D$. Consider an actual block

$$u=u_0v_1v_2^2,\quad A_j\leq Nv_j<2A_j\ (j=1,2),\quad Nu\leq H,\qquad R=A_1A_2,\quad M=\max(A_1,A_2).$$

The good ideals v_1, v_2 are squarefree and coprime, with their literal primary/ray representatives; all other good factors are the unit ideal. Let F be the actual indicator, including capacity and every row mask. In the remaining neighborhoods the first factor is nonunit, so the primitive row character is nonprincipal and $E_u=I_u$ there. For $X,Y\in\{A_U,C_U,\mu\}$ define the actual signed slice

$$S_1^{XY}=\frac{1}{D}\sum_u F(u)\sum_{\substack{m,n\ \text{squarefree,good}\\(m,n)=1}} X(m)Y(n)\nu(m)\overline{\nu(n)}\chi_m(u)\overline{\chi_n(u)}W(Nm/D)\overline{W(Nn/D)}. \tag{32}$$

All columns in this formula are the original annular columns, including raw zero sentinels. No positive-semidefinite claim about the coprime truncation is made.

Proposition 37.1 (A bound for the actual signed $h=1$ slice)**.** *Uniformly in the original cutoff and this actual block,*

$$|S_1^{XY}|\ll_{\nu,S,\epsilon}\|W\|_\infty^2(HD)^\epsilon\min\left\{R+\frac{DR}{M}+\frac{D^{2/3}R}{M^{1/3}},\ D+R^2\right\}. \tag{33}$$

The exact identity underlying the estimate is

$$\begin{aligned} DS_1^{XY}&=\sum_{\substack{d\ \text{squarefree}\\(d,S)=1}}\mu(d)\sum_{u\ \text{in the actual block}\atop(d,u)=1}T_d^X(u)\overline{T_d^Y(u)},\\ T_d^X(u)&=\sum_{\substack{a\ \text{squarefree}\\(a,dS)=1}}X(da)\nu(a)\chi_a(u)W(Nd\,Na/D). \end{aligned} \tag{34}$$

In particular $X(da)$ is not replaced by $X(d)X(a)$, and the mask $(d,u)=1$ remains.

The proof applies the accepted block sieves inside each positive row norm in (34), then explicitly loses the d signs in an absolute triangle. At $H=D^{3/4}$ the two remaining power-scale neighborhoods $A_1\asymp D^{5/12}, A_2\asymp D^{1/6}$ and $A_1\asymp D^{3/4}, A_2\asymp 1$, with feasible capacity constants, both retain power $7/6$. The desired $3/4$ still needs a signed saving of $5/12$. This proves a separate signed-slice bound; it does not improve the published global envelope or supply the missing cancellation.

There is also an exact arithmetic description of the two-pivot phase. Put $\delta = 1 + 2\omega$ and, at each good prime ideal p, set $j(p) = J_p(\chi_p, \chi_p^2)$. Extend j multiplicatively on squarefree columns. Let $\Psi(a)$ be the unique generator of an ideal prime to 2δ satisfying $\Psi(a) \equiv 1 \pmod{2\delta}$.

Proposition 37.2 (The actual Jacobi column twist)**.** *The map Ψ is an algebraic Hecke character with exact finite conductor (2δ), of norm 12, and principal convention $\Psi((\alpha)) = \alpha$ for $\alpha \equiv 1 \pmod{2\delta}$ (infinity type $(1,0)$). For good squarefree columns,*

$$(j(p)) = p, \quad -j(p) = \Psi(p), \qquad \mu(m)j(m) = \Psi(m). \tag{35}$$

Define $\vartheta(m) = \Psi(m)/\sqrt{Nm}$ and $\rho(m) = j(m)/\sqrt{Nm} = \mu(m)\vartheta(m)$ on those columns. In the high window the actual gauged coefficients are $b_A = (r_U - 1)\vartheta\nu W$, $b_C = r_U\vartheta\nu W$ and $b_\mu = \vartheta\nu W$.

This does not replace the original row kernel. Its compensating inverse gauge remains, so all ordinary coefficient/operator norms are unchanged. The new character has nonzero infinity type and is not silently covered by a theorem stated only for finite-order twists. The identities in (35) involving μ must not be extended to arbitrary nonsquarefree sixth factors.

A fresh fixed-profile example takes $\nu = 1$, S the primes above $2, 3$, literal unit one and primary row generators 1 modulo 3. It lies in an actual $A_6 = 1$ block at $D = 60000$, $U = 244$, $H = 3833$, $A_1 = 37$, $A_2 = 7$. All 20 rows of the stated literal class are nonprincipal. A selected q has exactly two legal column orientations, of norms 97051/96091 and its reversal. Their correlations in the ω basis are $(-4, -2)$ and $(-2, 2)$, with mean -3. Their selected contribution is $S_{1,q}^{AA} = -1/10000$; deleting the mean gives zero. The centered squared norm is 6, not zero, and the proposed variance-only Cauchy bound would incorrectly require $36 \le 12$. This refutes that centering rule in a fixed-profile row block, not the asymptotic target. Other q contributions remain. The coprime-kernel inspector shows the actual rows, raw zeros, two-column Gram matrix and compensated gauge; no training or arithmetic spectrum is inferred from that finite view.

38 The signed divisor mean and a complete-column laboratory

The preceding signed-slice estimate loses the sign of the common ideal divisor. We now retain that variable in an exact analytic reduction and in a complete finite computation. Neither supplies the missing 5/12 saving at $H = D^{3/4}$. The computations below use a new fixed profile; they do not fill in the uncomputed columns of the earlier broad-plateau profile.

Let $\mathcal{C}$ be original squarefree good ideal columns in $[D, 2D]$, and let $\mathcal{R}$ be an actual finite literal row block, with nonnegative multiplicities w_u. All shared-prime zeros of the original $\chi_m(u)$ remain. For arbitrary complex coefficient vectors define the linear-first form

$$DQ_1(x, y) = \sum_{u \in \mathcal{R}} w_u \sum_{\substack{m,n \in \mathcal{C} \\ (m,n)=1}} x_m \overline{y_n} \chi_m(u) \overline{\chi_n(u)}. \tag{36}$$

The original component is obtained with $x_m = X_U(m)\nu(m)W(Nm/D)$ and the corresponding y_m. On the nonprincipal row blocks used below the original residue term vanishes. Equation (36) is a raw coprime form on a generic block; principal rows would still require their separate residue assembly.

For every ideal divisor of an active column put

$$F_d(x; u) = \sum_{m \in \mathcal{C},\ d \mid m} x_m \chi_m(u), \qquad Z_d(x, y) = \sum_u w_u F_d(x; u) \overline{F_d(y; u)}.$$

The exact identity is

$$DQ_1(x,y) = \sum_d \mu(d) Z_d(x,y). \tag{37}$$

There is no extra $\mu(q)$: the vectors already contain the original coefficients. Full divisors $d = m$ are essential to cancel self-pairs. Divisors and squarefreeness refer to ideals, including distinct ideals with the same rational norm.

Proposition 38.1 (A weighted divisor-incidence lift)**.** *Suppose $\lambda_d \geq 0$ is an arbitrary-coefficient bound*

$$\sum_u w_u |F_d(x;u)|^2 \leq \lambda_d \sum_{m \in \mathcal{C},\ d \mid m} |x_m|^2.$$

With $B(m) = \sum_{d \mid m} \lambda_d$ and $\|x\|_B^2 = \sum_m B(m)|x_m|^2$, one has

$$|DQ_1(x,y)| \leq \|x\|_B \|y\|_B. \tag{38}$$

For a finite literal phase table one may take the exactly computed

$$\lambda_d = E_d := \sum_u w_u \#\{m \in \mathcal{C} : d \mid m,\ \chi_m(u) \neq 0\}.$$

This last choice is a Frobenius certificate, not a new analytic estimate. For $x = t + e$ it gives the finite perturbation bound

$$|Q_1(x,x) - Q_1(t,t)| \leq D^{-1}\big(2\|t\|_B \|e\|_B + \|e\|_B^2\big). \tag{39}$$

Proof. Insert ideal-divisor inversion in (36). Apply row Cauchy to each Z_d and weighted Cauchy to the resulting sum over d. Then $\sum_d \lambda_d \|x_d\|_2^2 = \|x\|_B^2$. The squared Frobenius norm of the actual F_d matrix is exactly E_d. Expand $Q_1(t+e, t+e) - Q_1(t,t)$ to obtain the last assertion. Positivity of the coprime-truncated matrix was never assumed. □

Using an accepted positive-row sieve bound for λ_d only recovers the old exponent up to ideal-divisor factors. It does not estimate the signed sum before the triangle in (37).

There is also a genuinely signed scalar expansion. Fix squarefree residual ideals a, b, which may overlap, and an actual row u. Keep the full deleted set $\mathcal{T} = S \cup \mathrm{supp}(abu)$ and set

$$\mathcal{Z}_{U;a,b,\mathcal{T}}(s) = \sum_{\substack{d \text{ squarefree} \\ (d,\mathcal{T})=1}} \frac{\mu(d) A_U(da) A_U(db)}{(Nd)^s}, \qquad \Re s > 1.$$

For four positive cutoff variables put $P_p = 1 - (Np)^{-v_1} - (Np)^{-v_2}$ and $Q_p = 1 - (Np)^{-w_1} - (Np)^{-w_2}$. The common-divisor Euler factor is $1 - (Np)^{-s} P_p Q_p$. Its first-order factorization has four numerator and five denominator zeta factors:

$$\prod_{p \notin \mathcal{T}} (1 - (Np)^{-s} P_p Q_p) = \frac{\prod_{i=1}^2 \zeta_K^{\mathcal{T}}(s+v_i) \zeta_K^{\mathcal{T}}(s+w_i)}{\zeta_K^{\mathcal{T}}(s) \prod_{i,j=1}^2 \zeta_K^{\mathcal{T}}(s+v_i+w_j)}\, \mathcal{H}_{\mathcal{T}}(s;v,w). \tag{40}$$

The correction is normally convergent at $\Re s \geq \frac{1}{2} + \eta$, $\Re v_i, \Re w_j \geq 0$, for each fixed $\eta > 0$. This says nothing about zeros or residues of the zeta ratio. Finite-height Perron kernels, with their explicit endpoint/error costs, recover the cutoffs by successive limits.

A direct finite cutoff expansion gives more explicit information:

$$\mathcal{Z}_{U;a,b,\mathcal{T}}(s) = \frac{\mathcal{N}_{U;a,b,\mathcal{T}}(s)}{\zeta_K^{\mathcal{T}}(s)}. \tag{41}$$

The numerator is a finite sum over eight nonzero joint short-divisor states, holomorphic on $\Re s > 0$. Uniformly for $\sigma \geq \sigma_0 > 0$,

$$|\mathcal{N}_{U;a,b,\mathcal{T}}(\sigma + i\tau)| \ll_{K,\sigma_0,\epsilon} (Na\,Nb)^{\epsilon} U^{4\max(1-\sigma,0)+\epsilon}.$$

The costs near $\sigma = 1/2$ persist. The remaining long state has the Möbius sign, but the eight finite short states cannot be omitted. A fixed-residual reciprocal zeta factor also does not prove an uncancelled pole in the complete coherent sum.

Exact complete-column reference. Choose once $\delta = 1/240000$ and $W_\delta(y) = s((y-1)/\delta)s((2-y)/\delta)$, where s is the smooth step from zero to one. For the finite grid $D \leq 60000$ every strict integer norm node has $W_\delta = 1$; endpoints vanish. Its derivatives cost $O_k(\delta^{-k})$. The three cases have different declared row blocks, not a common asymptotic scaling experiment. They use principal ν, S above $2, 3$, literal class one and nonprincipal actual rows.

D	columns	divisors	rows	S_1^{AA}
144	41	53	5	$-19/36$
576	164	212	11	$-221/288$
60000	16953	21750	20	$-5119/5000$

In the last case the exact even/odd Möbius masses are 543416 and 604844, giving numerator -61428 before division by D. All 339060 raw column–row values, including 47908 zeros, and all nine cross components are retained. In particular 1361 columns have nonsquarefree rational norm while remaining squarefree as ideals. The signed-divisor inspector exposes the actual states and dimensions. These finite negative slices are neither whole-family asymptotic counterexamples nor a positive arithmetic determinant or GRH result.

39 A target-sized budget for rough-prime masks

The actual signed divisor mean admits a useful quantitative simplification: the rough-prime coprimality mask can be restored with an explicit correction bound. The remaining main term still retains the Möbius sign and the coherent row mean; it is not estimated by this result.

Work over $\mathbb{Z}[\omega]$. Fix a finite S containing the primes above $2, 3$, a unitary multiplicative ν on good ideals, and one common complex profile W supported on $[1, 2]$, smooth and zero at both endpoints. Put $B = \|W\|_\infty$ and assume

$$1 \leq U, \qquad U^2 \leq D, \qquad 2D \leq 7U^2, \qquad H \geq 1. \tag{42}$$

Let $\mathcal{R}$ be the actual finite family of nonzero literal rows, with original weights $w_u \geq 0$, $1 \leq Nu \leq H$, and $R_w = \sum_u w_u$. All original internal squarefree/coprime conditions, unit and S-ray classes, capacity and sixth-factor masks remain in this family. Write S_1^{AA} for the raw form (36) with $x_m = A_U(m)\nu(m)W(Nm/D)$, where

$$A_U(g) = \sum_{\substack{xy|g,\ (x,y)=1 \\ Nx\leq U,\ Ny\leq U}} \mu(x)\mu(y). \tag{43}$$

Here the columns in (36) are the COMPLETE original good squarefree annulus $[D, 2D]$, with the common W as its norm cutoff and the original full-good-ideal/ν, S formulation. No selected-column indicator or additional column-ray filter is allowed in this reduction; all actual row unit/ray/capacity filters remain. Principal-row residues, if present, require their separate assembly. For the intended $A_1 > 1$ blocks the defining conductor of ν is supported in S, and the rows are nonprincipal.

Let f range over good squarefree U-smooth ideals, put $F = Nf$, and set

$$T_f(t;u) = \sum_{\substack{a \text{ good squarefree} \\ (a,f)=1}} A_U(fa)\nu(a)\chi_a(u)W(FtNa/D). \tag{44}$$

Here $\chi_a(u) = 0$ on $(a,u) > 1$. For a prime ideal p, let $T_f^{(p)}(t;u)$ impose the additional condition $(a,p) = 1$. The smooth factor f is distinct from the original column-pair gcd sector $h = 1$. Define the following *unnormalized* expressions:

$$K_0 = \sum_u w_u \sum_{\substack{f \text{ good SF, } U\text{-smooth} \\ (f,u)=1}} \mu(f)|T_f(1;u)|^2, \tag{45}$$

$$K_T = \sum_u w_u \sum_{\substack{f \text{ good SF, } U\text{-smooth} \\ (f,u)=1}} \mu(f) \sum_{\substack{p\notin S,\ Np>U \\ p\nmid u}} |T_f^{(p)}(Np;u)|^2, \tag{46}$$

$$K_S = \sum_u w_u \sum_{\substack{f \text{ good SF, } U\text{-smooth} \\ (f,u)=1}} \mu(f) \sum_{p\notin S,\ Np>U} |T_f(Np;u)|^2. \tag{47}$$

Throughout, p denotes a prime ideal. None of these expressions is declared positive, since the $\mu(f)$ signs remain.

The exact decomposition of the common ideal divisor $d = fe$ has at most one rough prime when the residual pair is nonunit. The unit residual pair also permits two distinct rough primes, necessarily with $f = 1$. Its exact correction is

$$C_2 = \sum_{\substack{\{p,q\} \text{ unordered, } p\neq q \\ p,q\notin S,\ Np,Nq>U}} |W(NpNq/D)|^2 \sum_{\substack{u\in\mathcal{R} \\ (u,pq)=1}} w_u. \tag{48}$$

Conjugate primes of equal rational norm are included. One has

$$DS_1^{AA} = K_0 - K_T + C_2. \tag{49}$$

In K_S, terms with $p \mid a$ or $p \mid u$ are explicitly auxiliary: fpa can fail to be squarefree. They use $A_U(fa)$ as written, not a new original column or a substitution for $A_U(fpa)$.

Theorem 39.1 (Explicit rough-prime mask budget)**.** *Put* $\kappa_U(u) = \#\{p \notin S : p \mid u,\ Np > U\}$ *and* $\Delta = (K_S - K_T)/D$. *Under* (42),

$$|\Delta| \le B^2\left(229376\sum_u w_u\kappa_U(u) + 2048R_w\right), \tag{50}$$

$$0 \le C_2/D \le 1792B^2R_w. \tag{51}$$

Consequently

$$S_1^{AA} = \frac{K_0 - K_S}{D} + E, \tag{52}$$

$$|E| \le B^2\left(229376\sum_u w_u\kappa_U(u) + 3840R_w\right). \tag{53}$$

If $H \le D^{3/4}$, then every row has $\kappa_U(u) \le 1$, and in particular

$$|E| \le 233216 B^2 R_w \le 2^{18} B^2 R_w. \tag{54}$$

Only the rough-prime deletion has been restored; the smooth-divisor mask, literal row family and original coefficients remain. The explicit bound uses the exact restricted identity

$$A_U(g) = \mu(g)(2\mathbf{1}_{Ng \le U} - 1) \quad (Ng \le 7U,\ g \text{ good squarefree}).$$

Every active term of $T_f(Np; u)$ has $N(fa) < 7U$, hence $|A_U(fa)| = 1$ there. This is not a shrinkage or proxy for the whole original coefficient family. Only absolute correction bounds use it; the signed coefficients in the clean main remain unchanged.

For general H, one has $\kappa_U(u) \le 2\log(2H)/\log(2U)$, also at $U = 1$. For an actual two-pivot block with $R_w \ll A_1 A_2 \le H$ and $H \le D^{3/4}$, (54) gives an $O(B^2 H)$ error with no divisor-power or profile-derivative loss. The remaining target problem for this raw A–A slice is the clean signed main $(K_0 - K_S)/D$. It has not been bounded here; the 7/6 envelope and missing 5/12 improvement for the complete original remainder are unchanged. The argument neither deletes a coherent mean nor treats the coprime-truncated form as positive.

Three exact finite controls. These use principal ν, S above $2, 3$, literal unit class one and complete original good squarefree annuli. The first two row variables are squarefree, good and coprime and may include the unit ideal; all other factors equal one.

D	U	H	A_1	A_2	columns	rows
200	10	53	13	1	52	5
512	16	107	19	1	139	7
900	30	384	7	7	250	2

Put $\eta_0(t) = \exp(-1/t)$ for $t > 0$ and $\eta_0(t) = 0$ for $t \le 0$, and define

$$s(t) = \frac{\eta_0(t)}{\eta_0(t) + \eta_0(1 - t)}.$$

Choose once $\delta = 1/4096$ and $W_\delta(t) = s((t-1)/\delta)s((2-t)/\delta)$. For integer $D \le 1024$, strict integer norm nodes have weight one and endpoints have weight zero; its derivatives cost $O_k(4096^k)$. The following numbers are RAW, before division by D:

D	K_0	K_S	K_T	E_{row}	E_{res}	C_2	DS_1^{AA}
200	339	235	209	20	6	21	151
512	366	781	711	48	22	89	−256
900	436	410	366	0	44	28	98

In every case $K_S - K_T = E_{\text{row}} + E_{\text{res}}$ and $DS_1^{AA} = K_0 - K_T + C_2$. The displayed nonnegative values of $E_{\text{row}}, E_{\text{res}}$ are finite observations; no general nonnegativity is assumed for either signed correction. Only C_2 is nonnegative by definition. The first two capacities are $\lfloor D^{3/4} \rfloor$. The third $H = 384$ is a declared noncritical fixture capacity, not $\lfloor 900^{3/4} \rfloor$; both of its rows have $\kappa_U = 0$. All ideal and raw-zero masks, unit residual states and explicitly auxiliary non-squarefree p^2 terms are retained. The exact control summary records the scopes and exact fractions after one division by D. These are finite identities, not a proof or numerical fit of the analytic bounds, and do not estimate the signed clean main.

40 The coherent signed main and its remaining inputs

The physical sixth-power-free comparison in Section 41 provides a new positive upper bound. We now collect the exact coherent reductions that it can enter. They preserve the original mean and all raw masks; none proves the target by a scalar cancellation assertion.

Use the complete original good squarefree annulus and common complex profile of Section 39. Write $\mathcal{U}$ for the actual nonzero literal rows, $w_u \ge 0$, $R_w = \sum_u w_u$ and $B = \|W\|_\infty$. Retain fixed S, ν, all internal row masks and (42). Every identity below is a raw A–A identity; principal residues still require their separate assembly. Classical operator specializations are only for the original unweighted $A_6 = 1$ rows, or their already justified bounded natural multiplicities.

First there is a useful distinction between two auxiliary kernels. Let K_{all} be K_S of (47) with the U-smooth restriction on its squarefree f removed, retaining $(f,u)=1$ and exactly $A_U(fa)$ in every T_f. Set

$$\begin{aligned} P_2 &= \sum_u w_u \sum_{\substack{p\notin S\\ Np>U}} \sum_{\substack{q\notin S,\ Nq>U\\ q\nmid u}} |W(NpNq/D)|^2, \\ K_S &= K_{\mathrm{all}} + P_2, \qquad 0 \le P_2/D \le 3584 B^2 R_w. \end{aligned} \tag{55}$$

Here (p,q) is ordered, $p=q$ is allowed, and only q has the stated row mask. This is distinct from the original unordered, distinct-prime C_2 in (48). In particular

$$S_1^{AA} = \frac{K_0 - K_{\mathrm{all}}}{D} + \Delta + \frac{C_2 - P_2}{D}. \tag{56}$$

At $H \le D^{3/4}$ the error in this *new* auxiliary main is at most $236800B^2R_w$. The old $233216B^2R_w$ bound for $(K_0 - K_S)/D$ is unchanged. The physical comparison in (64) estimates the new combined main, in its stated $A_6 = 1$ scope, without estimating its two summands separately.

The prime-discrepancy clauses below additionally require $U \ge 7$. Define

$$E_S(t) = \pi_K^S(t) - \mathrm{Li}_2(t), \qquad \mathrm{Li}_2(t) = \int_2^t \frac{dv}{\log v}, \qquad M_1(W) = \max\{B, \|xW'(x)\|_\infty\}.$$

The count is of prime ideals outside fixed S, including distinct conjugate ideals and inert prime norms. Put

$$Q(t) = \sum_{\substack{f \text{ good SF}\\ f\ U\text{-smooth}}} \mu(f) \sum_{\substack{u\in\mathcal{U}\\ (f,u)=1}} w_u |T_f(t;u)|^2, \qquad \mathcal{J} = \int_U^\infty \frac{Q(t)}{\log t}\, dt.$$

Proposition 40.1 (A joint form retaining its mean)**.** *On the finite space of good squarefree g with $Ng \le 2D/U$, let*

$$b_g(t) = A_U(g)\nu(g)W(Ng\,t/D), \quad (\mathsf{R}c)(u) = \sqrt{w_u}\sum_g c_g\chi_g(u), \quad (\mathsf{P}_f c)_g = \mathbf{1}_{f|g}c_g.$$

With ordinary counting ℓ^2 norms,

$$Q(t) = b(t)^*\mathcal{M}_U b(t), \qquad \mathcal{M}_U = \sum_{f\ U\text{-smooth}} \mu(f)\mathsf{P}_f\mathsf{R}^*\mathsf{R}\mathsf{P}_f, \quad t \ge U. \tag{57}$$

The matrix is self-adjoint, with no positivity assertion. The exact prime substitution is

$$K_S = \mathcal{J} + \mathcal{E}_{\mathrm{PIT}}, \qquad \mathcal{E}_{\mathrm{PIT}} = -E_S(U)Q(U) - \int_U^\infty E_S(t)Q'(t)\, dt. \tag{58}$$

The endpoint excludes $Np = U$ exactly. It must remain.

For any true arbitrary-coefficient row norm $\lambda_*(L)$, use the direct unit-column value R_w on $1/2 \le L < 1$. On the active range $1 \le L \le D/U$, the current original two-pivot bound is the minimum in Proposition 41.4; it is not a theorem for arbitrary row weights. Let $\lambda^{\#}(D/T)$ dominate this norm on the two adjacent active length dyads. Then

$$\frac{|\mathcal{E}_{\mathrm{PIT}}|}{D} \ll_\epsilon (HD)^\epsilon M_1(W)^2 \left\{ \frac{|E_S(U)|}{U} \lambda_*(D/U) + \sum_T \mathfrak{d}_2(T) \lambda^{\#}(D/T) \right\}, \tag{59}$$

where $T = 2^j U < 2D$ and $\mathfrak{d}_2(T)^2 = \int_T^{2T} |E_S(t)|^2 t^{-3}\, dt$. This is a sufficient-input budget, not an asserted power estimate for $\mathfrak{d}_2$ or for the endpoint. The pointwise version retains $\lambda_*(D/U) + \int_{1/2}^{D/U} \lambda_*(L)\, dL/L$, with the current minimum inside that integral.

On good rough-prime columns the exact quadratic-form compression is $\Pi_{\mathcal{P}} \mathcal{M}_U \Pi_{\mathcal{P}} = \Pi_{\mathcal{P}} \mathsf{R}^* \mathsf{R} \Pi_{\mathcal{P}}$. This is not an invariant-subspace assertion: off-block couplings remain and can cancel its contribution. Radial Mellin Parseval likewise retains equal-norm ideal correlations, including their derivatives; it neither orthogonalizes orientations nor permits deleting their mean. Even exact prime density leaves $K_0 - \mathcal{J}$ to estimate.

Proposition 40.2 (A sufficient smooth-divisor tail cutoff)**.** *For the original unweighted $A_6 = 1$ two-pivot rows put $R = A_1 A_2$, $M_{\mathrm{row}} = \max(A_1, A_2)$. Restrict the smooth f sums in K_0, K_S to $Nf > Y \ge 1$. The inherited positive input gives*

$$\begin{aligned} \frac{|K_0^{>Y}|}{D} &\ll_\epsilon (HD)^\epsilon B^2 \left\{ R + \frac{DR}{M_{\mathrm{row}} Y} + \frac{D^{2/3} R}{M_{\mathrm{row}}^{1/3} Y^{2/3}} \right\}, \\ K_S^{>Y} &= 0 \qquad (Y \ge 2D/U). \end{aligned} \tag{60}$$

If $R \ll H$, a sufficient target-size tail cutoff is

$$Y \ge \max\left\{ 1, \frac{2D}{U}, \frac{DR}{M_{\mathrm{row}} H}, \frac{DR^{3/2}}{M_{\mathrm{row}}^{1/2} H^{3/2}} \right\}.$$

This is a coarse inherited sufficient bound, not signed cancellation or an optimal localization. The smooth incidence divisor f is not the original column-pair gcd h.

Theorem 40.3 (A conditional input diagnostic)**.** *Fix S and $0 < \alpha < 1/2$. If for every $\epsilon > 0$,*

$$\mathfrak{d}_2(T) \ll_{\epsilon, K, S} T^{-\alpha + \epsilon}$$

holds for all sufficiently large real T, then

$$\zeta_K(s) \ne 0 \qquad (\mathrm{Re}\, s > \max\{1 - \alpha, 1/2\},\ s \ne 1).$$

This is a strict-region implication from an explicitly hypothetical input, not a new zero-free theorem or a necessary condition for the actual signed target.

With the current physical input the pure witness has the sufficient PIT majorant 19/24, and the mixed witness retains the smaller 7/9 from the minimum. Independent scalar powers $U^{-1/12}$ and $U^{-1/18}$ would suffice for those majorants, with compatible endpoint control; their all-scale mean-square versions would entail the strict strips 11/12 and 17/18 above. None is asserted available. The old pure 5/6 budget and $U^{-1/6}$ input, with conditional strip 5/6, are historical values for the former operator bound. Proposition 41.2 gives the current global 9/8 comparison, and Proposition 41.3 the scoped $A_6 = 1$ comparison 13/12. The target, principal/short assembly beyond the stated scopes, independent determinant/zero-location bridges and full analytic/category Lean formalization remain open.

41 A sieve comparison indexed by the physical radicand

This section applies a new external preprint input to the original rows. It is an application of that theorem, not a new proof of a large sieve, a signed target estimate, or an analytic Lean result. Throughout, $K = \mathbb{Q}(\omega)$, S is fixed and contains the primes above $2, 3$ and the original conductor exclusions, and ν is the original fixed finite-order Hecke column character. Let $D, H \geq 1$, $U \geq 1$, $U^2 \leq D$, and let W be one common possibly complex smooth profile supported on $[1, 2]$, zero at its endpoints. Write $B_W = \|W\|_\infty$. The estimates below concern the original unweighted literal rows, or already justified bounded natural multiplicities. They do not insert arbitrary positive row weights into an unweighted theorem.

Pinned external input. De Faveri's arXiv:2610.04045v1, Theorem 1.1, bounds the squared row norm $\Theta_n(A, B)$ for the Section 2.3 family of order-dividing-n Hecke characters, with both indices nth-power-free ideals prime to fixed S:

$$\Theta_n(A, B) \ll_\epsilon (AB)^\epsilon (A + B + A^{1-1/n} B^{2/n} + A^{2/n} B^{1-1/n}). \tag{61}$$

Here $n \geq 3$, K contains μ_n, and the chosen fixed S-integer ring has class number one. The constants may depend on the fixed family construction. We use $n = 6$, not a conjectural $A + B$ bound; the additional μ_{2n} condition of that paper's Theorem 1.4 is not a hypothesis of Theorem 1.1. Set

$$\Phi_6(A, B) = A + B + A^{5/6} B^{1/3} + A^{1/3} B^{5/6},$$
$$\Phi_{\mathrm{all}}(H, D) = H + DH^{1/6} + H^{5/6} D^{1/3} + H^{1/3} D^{5/6}.$$

Proposition 41.1 (Exact raw-family identification)**.** *There is an allowed fixed choice of the data in* (61) *for which, for every good ideal* $\mathfrak{a}$*, with unique generator* $\alpha_\mathfrak{a} \equiv 1 \pmod 3$*,*

$$\chi_\mathfrak{a}(\mathfrak{m}) = \left(\frac{\alpha_\mathfrak{a}}{\mathfrak{m}}\right)_6$$

including the original zero at each common prime. In particular, the row index is the physical sixth-power-free ideal, including valuations $2, 3, 4, 5$*, rather than its radical conductor.*

The extended proof specifies the data and removes the possible unit ambiguity. No extra column-ray selection is made. For the unique outside-S decomposition of an actual literal row,

$$\begin{aligned} u &= u_0 \alpha_\mathfrak{a} \alpha_{\mathfrak{r}_6}^6, & \mathfrak{a} &= \mathfrak{r}_1 \mathfrak{r}_2^2 \mathfrak{r}_3^3 \mathfrak{r}_4^4 \mathfrak{r}_5^5, \\ \chi_\mathfrak{m}(u) &= \chi_\mathfrak{m}(u_0) \chi_\mathfrak{a}(\mathfrak{m}) \mathbf{1}_{(\mathfrak{m}, \mathfrak{r}_6) = 1}, & Nu &= Nu_0 \, N\mathfrak{a} \, (N\mathfrak{r}_6)^6. \end{aligned} \tag{62}$$

The first five factors are squarefree and pairwise coprime; $\mathfrak{r}_6$ is arbitrary and may overlap them. All original union, common-prime, literal, ray and capacity restrictions remain.

Proposition 41.2 (Physical-row comparison)**.** *For the original raw sums* I_u *of* (13)*, and for the original residue-subtracted* $E_u = I_u - P_u$ *with fixed* ν, S*,*

$$\sum_{0 < Nu \leq H} |I_u|^2 + \sum_{0 < Nu \leq H} |E_u|^2 \ll_{\nu, S, \epsilon} B_W^2 (HD)^\epsilon \Phi_{\mathrm{all}}(H, D).$$

For $1 \leq H \leq D$ *one has*

$$\Phi_{\mathrm{all}}(H, D) \leq 4DH^{1/6}. \tag{63}$$

Thus the critical positive envelope at $H = D^{3/4}$ *is* $D^{9/8+\epsilon}$*, improving the previous* $7/6$ *comparison by* $1/24$ *while still exceeding the* $3/4$ *target by* $3/8$*.*

The agreement with the varying-profile obstruction of Section 36 concerns only a supremum-norm-controlled class allowing W_D to vary. It is not a lower bound or a sharpness assertion for one fixed W, nor a GRH counterexample.

Proposition 41.3 (The actual $A_6 = 1$ signed slice)**.** *For an original unweighted two-pivot block, retaining all its actual filters and the complete annular columns, the raw form S_1^{AA} of* (32) *satisfies*

$$|S_1^{AA}| \ll_{\nu,S,\epsilon} B_W^2 (HD)^{\epsilon} \Phi_6(H, D).$$

The same bound holds after summing the fixed-S parts of such rows: the D term costs a fixed polylogarithmic count, absorbed in $(HD)^{\epsilon}$, not a convergent exponent-zero Euler sum. At $H = D^{3/4}$ the available power is $13/12$*, an improvement of* $1/12$ *from the old* $7/6$ *majorant, but still* $1/3$ *above the target.*

The proof inserts (61) into the exact original common-divisor identity (34); $A_U(dn)$ is not factored and is not put into the rough short-coefficient domain. Under the additional high-window and capacity hypotheses of Section 40, its exact ordered rough correction gives

$$\left| \frac{K_0 - K_{\mathrm{all}}}{D} \right| \le |S_1^{AA}| + 236800 B_W^2 R_w. \tag{64}$$

This estimates the actual combined form through its exact identity; it does not take separate main-term triangles in K_0, K_{all}, and does not prove signed cancellation at the target scale.

Proposition 41.4 (Updated short positive certificate)**.** *On an original unweighted block, put $R = A_1 A_2$, $M = \max(A_1, A_2)$ and $H_b = \min\{H/Nu_0, 8A_1A_2^2\}$ for fixed literal/S factor u_0. Empty blocks contribute zero; a nonempty block has $H_b \ge 1$. For $1 \le L \le D$ the actual squared row-operator norm has the bound*

$$\lambda_{\mathrm{now}}(L) \ll (HD)^{\epsilon} \min \left\{ \begin{matrix} R + \dfrac{LR}{M} + \dfrac{L^{2/3}R}{M^{1/3}}, \quad L + R^2, \\ \Phi_6(H_b, L) \end{matrix} \right\}.$$

For $1/2 \le L < 1$ use the direct unit-column norm R_w. This new norm may be substituted into the mean-retaining joint Abel criterion of Proposition 40.1, with its lower endpoint intact. At $H = D^{3/4}$, $U \asymp D^{1/2}$, its pure witness costs $19/24$*; at the displayed mixed witness the minimum retains the old* $7/9$ *bound, which beats the generic* $19/24$*.*

In this positive certificate sufficient independent scalar discrepancy powers are respectively

$$U^{-1/12+\epsilon} \quad \text{and} \quad U^{-1/18+\epsilon}. \tag{65}$$

If imposed as all-scale mean-square inputs, these would entail the conditional zero-free strips $\Re s > 11/12$ and $\Re s > 17/18$ of Theorem 40.3. Neither input nor strip is proved here; they are not necessary conditions for the actual signed target. The endpoint and coherent $K_0 - J$ remain. Smooth-tail localization (Proposition 40.2) does not remove $f = 1$ or supply this estimate.

42 Coherent dual bands, actual cutoffs and arithmetic localization

The physical-row comparison in Section 41 leaves a signed problem. We now retain its exact primitive roots and sharp row capacity, prove quantitative finite-prefix and divisor-band comparisons, and localize a coherent sector that can already be controlled. The remaining unequal-norm signed moment is stated explicitly. These results do not improve the whole signed power $13/12$ or the global unsigned power $9/8$. The deep sieve input remains the pinned external theorem (61); its cubic specialization is used below.

42.1 The original form and the scope of the dual representation

Let $K=\mathbb{Q}(\omega)$, $\mathcal{O}=\mathbb{Z}[\omega]$ and $\delta=1+2\omega=i\sqrt{3}$. The fixed set S contains the primes above $2,3$ and the original exclusions. All columns are the complete good squarefree *ideal* annulus $D\le N\mathfrak{m}\le 2D$, $D>1$. Distinct ideals of the same norm remain distinct. Put

$$\begin{aligned} A_U(\mathfrak{m}) &= \sum_{\substack{\mathfrak{r}\mathfrak{s}\mid\mathfrak{m}\\(\mathfrak{r},\mathfrak{s})=1\\ N\mathfrak{r},N\mathfrak{s}\le U}} \mu(\mathfrak{r})\mu(\mathfrak{s}),\\ c_{\mathfrak{m}} &= A_U(\mathfrak{m})\nu(\mathfrak{m})R_{\mathfrak{m}}(u_0)W(N\mathfrak{m}/D). \end{aligned} \tag{66}$$

Here $1\le U\le\sqrt{D}$, ν is the original fixed unitary finite-order column character, and W is one common possibly complex smooth profile supported on $[1,2]$, zero at its endpoints. Write $B_W=\|W\|_\infty$. For a good ideal $\mathfrak{v}$, let $\alpha_{\mathfrak{v}}\equiv 1 \pmod 3$ be its unique primary generator and set $R_{\mathfrak{m}}(\mathfrak{v})=(\alpha_{\mathfrak{v}}/\mathfrak{m})_6$. The literal symbol is zero at a common prime. A primitive extension at an S-argument below is a different object, not restoration of a deleted raw row value.

Let $A_1,A_2\ge 1$. Only the original unweighted two-pivot rows are used:

$$u=u_0\alpha_{\mathfrak{v}_1}\alpha_{\mathfrak{v}_2}^2,\qquad A_i\le N\mathfrak{v}_i<2A_i,\qquad Nu_0N\mathfrak{v}_1(N\mathfrak{v}_2)^2\le H.$$

The ideals $\mathfrak{v}_1,\mathfrak{v}_2$ are good, squarefree and coprime; all actual literal and ray restrictions remain. The other four factors are the unit ideal. The fixed u_0 is literal unit/S-supported, without good-prime support. Extra good frozen factors, a sum over varying u_0, and arbitrary row weights are not included. Write $R=A_1A_2$ and let R_w be the actual number of rows. The normalization is

$$\begin{aligned} \mathcal{P}_D &= DS_1^{AA}\\ &= \sum_{(\mathfrak{v}_1,\mathfrak{v}_2)\ \text{actual}}\ \sum_{(\mathfrak{m},\mathfrak{n})=1} c_{\mathfrak{m}}\overline{c_{\mathfrak{n}}}R_{\mathfrak{m}}(\mathfrak{v}_1\mathfrak{v}_2^2)\overline{R_{\mathfrak{n}}(\mathfrak{v}_1\mathfrak{v}_2^2)}. \end{aligned} \tag{67}$$

The original Möbius signs are already in the two A_U coefficients; no additional $\mu(\mathfrak{m}\mathfrak{n})$ is inserted. This raw coprime form is not a positive energy or principal-term assembly.

42.2 Primitive normalization and an exact finite-cutoff kernel

For each source class A, use the fixed decomposition $\widetilde{\chi}_{\mathfrak{m}}=\xi_{\mathfrak{m}}\psi_A$, $\mathrm{cond}(\xi_{\mathfrak{m}})=\mathfrak{m}$, and $\beta_{\mathfrak{m}}=\epsilon(\xi_{\mathfrak{m}})\psi_A(\mathfrak{m})$. Individual ξ and ψ may have nonzero fixed angular and radial unitary types; they are not declared sixth-order. The finite source reciprocity factors and the *whole* original row-ray projector are expanded in their complete finite character basis, with coefficients $h_{AB\rho}(\mathfrak{v}_2)$. On coprime class-$A,B$ columns the resulting primitive character is

$$\begin{aligned} &\sigma=\mathrm{prim}(\rho\psi_A\overline{\psi_B}),\quad \mathfrak{f}_S=\mathrm{cond}(\sigma),\\ &\eta=\xi_{\mathfrak{m}}\overline{\xi_{\mathfrak{n}}}\sigma,\qquad\qquad F=N\mathfrak{f}_S\,N\mathfrak{m}\,N\mathfrak{n}. \end{aligned} \tag{68}$$

Its total infinity type is zero. The actual S conductor, including any drop, is retained.

Proposition 42.1 (Primitive roots with their types retained)**.** *For a primitive unitary Hecke character $\chi((\alpha))=\theta(\alpha)\,\mathrm{phase}(\alpha)^kN(\alpha)^{it}$, with conductor (γ), let*

$$\tau_\gamma(\theta)=\sum_{x \bmod \gamma}\theta(x)\exp\bigl(2\pi i\,\mathrm{Tr}(x/(\delta\gamma))\bigr).$$

Then

$$\epsilon(\chi) = (3N\mathfrak{f})^{it} i^{-|k|} \operatorname{phase}(\delta\gamma)^k \frac{\tau_\gamma(\theta)}{\sqrt{N\mathfrak{f}}}. \tag{69}$$

The unit-conductor/principal formula has its usual meromorphic terms. For pairwise coprime primitive conductors,

$$\epsilon\Big(\prod_j \chi_j\Big) = i^{\sum_j |k_j| - |\sum_j k_j|} \prod_j \epsilon(\chi_j) \prod_{i\neq j} \chi_i(\mathfrak{f}_j). \tag{70}$$

Consequently the actual character in (68) *satisfies*

$$\begin{aligned} \epsilon(\eta) &= \kappa_{AB\rho} u_{\mathfrak{m}} \overline{v_{\mathfrak{n}}}, \\ u_{\mathfrak{m}} &= \beta_{\mathfrak{m}} \xi_{\mathfrak{m}}(\mathfrak{f}_S) \rho(\mathfrak{m}), \\ v_{\mathfrak{n}} &= \beta_{\mathfrak{n}} \xi_{\mathfrak{n}}(\mathfrak{f}_S) \overline{\rho(\mathfrak{n})}. \end{aligned} \tag{71}$$

The scalar $\kappa_{AB\rho}$ is fixed. These are two distinct unitary column gauges. The primitive values $\sigma(\mathfrak{e})$ and $\sigma(\mathfrak{h})$ at S, including zero, remain separately in the kernel; they cannot be replaced by ρ there.

This closes the normalization hypothesis for these stated conventions. A bounded, invertible change of phases gives no power saving by itself.

For a fixed actual $\mathfrak{v}_2$, define the exact integer endpoints

$$a = \lceil A_1 \rceil, \qquad b = \min\{\lceil 2A_1 \rceil - 1, \lfloor H/(Nu_0(N\mathfrak{v}_2)^2) \rfloor\}.$$

An empty interval contributes zero. Choose one fixed nondecreasing smooth step s_0, equal to zero on $(-\infty, 0]$ and one on $[1, \infty)$, and put

$$G_{\mathfrak{v}_2}(x) = s_0(4(A_1 x - a + 1/2)) s_0(4(b + 1/2 - A_1 x)). \tag{72}$$

It is the exact row indicator at every ideal norm node, has amplitude at most one, support in $[1/2, 5/2]$ and total variation at most two. Its derivatives cost powers of A_1; it is not the fixed column W. Set $C_0 = 3/(4\pi^2)$ and

$$\dot{G}(X) = C_0^{-1/2} \int_0^\infty G(t) J_0(2\sqrt{Xt/C_0})\, dt. \tag{73}$$

Proposition 42.2 (The actual signed kernel before absolute norms)**.** *For every nonzero class/ray component, the good conductor of η is $\mathfrak{mn}$, so η is nonprincipal. Put $P_S = \prod_{p\in S} p$ and*

$$L_{\mathfrak{e}} = \frac{A_1}{(N\mathfrak{k})^2 N\mathfrak{l} N\mathfrak{e}}, \qquad K_{\mathfrak{l}} = \sqrt{b/N\mathfrak{l}}.$$

Squarefree-row inversion, finite S deletion and primitive Poisson give (67) *by summing the following kernel:*

$$\begin{aligned} &c_{\mathfrak{m}} \overline{c_{\mathfrak{n}}} R_{\mathfrak{m}}(\mathfrak{v}_2^2) \overline{R_{\mathfrak{n}}(\mathfrak{v}_2^2)} h_{AB\rho}(\mathfrak{v}_2) \\ &\quad \cdot \mu(\mathfrak{k})\mu(\mathfrak{l})\mu(\mathfrak{e}) \eta(\mathfrak{k}^2\mathfrak{l}\mathfrak{e}) \epsilon(\eta) \frac{L_{\mathfrak{e}}}{\sqrt{F}} \sum_{\mathfrak{y}} \dot{G}_{\mathfrak{v}_2}(L_{\mathfrak{e}} N\mathfrak{y}/F) \overline{\eta(\mathfrak{y})}. \end{aligned} \tag{74}$$

The outer sums are over every actual $\mathfrak{v}_2$, every class/ray component and original coprime column pair, all $\mathfrak{l} \mid \mathfrak{v}_2$ with $N\mathfrak{l} \le b$, all good squarefree $\mathfrak{k}$ with $(\mathfrak{k}, \mathfrak{v}_2) = 1$ and $N\mathfrak{k} \le K_{\mathfrak{l}}$, and every $\mathfrak{e} \mid P_S$. The dual sum is over all *nonzero integral ideals, including S-supported ones. It converges at each fixed parameter set with the actual derivative constants of G. The original pre-S cutoff is never extended in this identity.*

All signs, means, product norms and raw zeros occur before a triangle. A higher induced modulus does not have a unit Gauss root automatically: its nonprimitive additive frequencies may vanish.

42.3 What finite prefixes and actual divisor bands now satisfy

Write

$$\begin{aligned}\Phi(Z,L) &= Z + LZ^{1/6} + Z^{5/6}L^{1/3} + Z^{1/3}L^{5/6},\\ I(t) &= \sum_{j\ge 0}\frac{t^j}{(j!)^2}.\end{aligned} \tag{75}$$

Let $\mathcal{P}_{\le X}$ retain all terms of (74) with $N\mathfrak{y} \le X$, including every source/ray/S component.

Proposition 42.3 (All-component exact finite prefixes)**.** *For every finite $X \ge 1$,*

$$\frac{|\mathcal{P}_{\le X}|}{D} \ll B_W^2(DXA_2)^{\epsilon}\frac{A_1A_2}{D}\Phi(X,D). \tag{76}$$

A separate exact all-orders estimate is

$$|\mathcal{P}_{\le X}| \ll B_W^2(DXA_2)^{\epsilon}A_1A_2\Phi(X,D)I\left(\frac{(5/2)A_1X}{C_0D^2}\right). \tag{77}$$

No completed Euler model, low-frequency approximation or $X \le \sqrt{H}$ hypothesis is used in (76)*. On a fixed same-source-class/$\rho = 1$ component, the additional critical-half band estimate is*

$$\frac{|\mathcal{P}_{C,1}(Y,2Y]|}{D} \ll B_W^2(DYA_2)^{\epsilon}A_2\sqrt{A_1/Y}\,\Phi(Y,D)(1+\log(2A_1))^2. \tag{78}$$

Constants depend on the fixed S, ray/source data, ν, step and ϵ, not on any hidden uniform smoothness of the sharp G.

At $H \asymp D^{3/4}$, the mixed witness $(A_1,A_2) \asymp (D^{5/12},D^{1/6})$ has a target-sized full finite prefix $X \asymp D$. The pure prefix at $X = D$ has upper power $11/12$. At $X = D^{3/8}$ the pure/mixed powers are $13/16$ and $31/48$. These growing bounds cannot be sent to $X = \infty$.

Proposition 42.4 (Finite transfer of the actual cutoff and profile)**.** *Let $X \ge 2$, $1 \le U \le X$, and let $K = (K_{\mathfrak{d}\mathfrak{a}})$ be a finite matrix indexed by good squarefree ideals of norms at most X, zero unless $(\mathfrak{d},\mathfrak{a}) = 1$ and $N\mathfrak{d}N\mathfrak{a} \le X$. All additional masks may remain in K. Set*

$$\begin{aligned}G_{\mathcal{I}\mathcal{J}} &= \max_{\mathfrak{d}\in\mathcal{I}} 3^{\omega(\mathfrak{d})}\max_{\mathfrak{a}\in\mathcal{J}} 3^{\omega(\mathfrak{a})},\\ c_P &= 1/\log(4X), \qquad L_T = \operatorname{asinh}(T/c_P)/\pi.\end{aligned}$$

Empty matrices are zero. For every finite $T \ge 8X$,

$$\|[A_U(\mathfrak{d}\mathfrak{a})K_{\mathfrak{d}\mathfrak{a}}]\|_{\mathrm{op}} \le G_{\mathcal{I}\mathcal{J}}\left(e^2L_T^2 + \frac{32X^{3/2}}{T}\right)\|K\|_{\mathrm{op}}. \tag{79}$$

For $T = (4X)^4$ this is $O_\epsilon(X^\epsilon\log^2(4X))\|K\|$. The common complex multiplier $W(N\mathfrak{d}N\mathfrak{a}/D)$ costs

$$\begin{aligned}M(W) &= \frac{1}{2\pi}\int_{\mathbb{R}}|\widehat{W}(it)|\,dt, \qquad \widehat{W}(s) = \int_0^\infty W(x)x^s\,\frac{dx}{x},\\ M(W) &\le \frac{\log 2}{\pi}(B_W + \|xW'(x) + x^2W''(x)\|_\infty).\end{aligned} \tag{80}$$

Two occurrences in a bilinear/Gram estimate cost the square of the transfer norm. This is not $A_U(\mathfrak{d}\mathfrak{a}) = A_U(\mathfrak{d})A_U(\mathfrak{a})$ or coefficient shrinkage.

Proposition 42.5 (A cubic comparison for an actual auxiliary band)**.** *Fix one same-source-class/$\rho = 1$ component of the auxiliary individual-column extension of* (74)*. Insert the exact coprime identity $\sum_{\mathfrak{d}|(\mathfrak{m},\mathfrak{n})}\mu(\mathfrak{d})$, and retain the* identity-source-class *divisors $E < N\mathfrak{d} \le 2E$, $1 \le E \le D/2$. Denote the resulting exact signed dual band by $\mathcal{P}^{\mathrm{id}}_{E,C}(Y,2Y]$. Put $L = D/E$ and $\Theta_3(E,L) = E + L + (EL)^{2/3}$. For fixed $0 < \lambda < 1/2$,*

$$\begin{aligned}\frac{|\mathcal{P}^{\mathrm{id}}_{E,C}(Y,2Y]|}{D} &\ll M(W)^2(DYA_2)^{\epsilon}A_2A_1^{1-\lambda}D^{2\lambda-1}Y^{-\lambda}\\ &\quad\cdot\min\{\Phi(Y,L),\ (Y/E)\Theta_3(E,L)\}.\end{aligned} \tag{81}$$

The arithmetic identity behind the new branch is

$$\beta_{\mathfrak{d}\mathfrak{a}} = \epsilon(\widetilde{\chi}_{\mathfrak{d}})\beta_{\mathfrak{a}}\alpha(1,C)\widetilde{\chi}_{\mathfrak{a}}(\mathfrak{d})^2, \qquad [\mathfrak{d}] = 1, \quad [\mathfrak{a}] = C. \tag{82}$$

It exposes the physical cubic residue matrix in $\mathfrak{d},\mathfrak{a}$. The external input is de Faveri, Theorem 1.1 and Example 2.2, now at $n = 3$, not a conductor-parametric sieve.

This improves a *scoped norm comparator*: at $E \asymp D^{2/3}$ the second branch is $O(Y)$. It still uses $|\mu(\mathfrak{d})| \le 1$ at its final Cauchy step. Such $\mathfrak{d}$ are auxiliary divisors, not the original column-pair gcd h. All divisor classes, small $\mathfrak{d}$ and the full high-dual sum are not bounded here.

42.4 Coherent means that can be localized, and those that remain

Proposition 42.6 (The full equal-norm sector)**.** *Original coprime squarefree columns with $N\mathfrak{m} = N\mathfrak{n} = N > 1$ satisfy $\mathfrak{n} = \overline{\mathfrak{m}}$, with rational-squarefree N supported on split rational primes. The original good masks require* both *primes above each such rational prime to avoid S; S need not be enlarged or conjugation-stable. There are $O(D)$ ordered annular pairs. Define S_{eq} as the normalized contribution to S_1^{AA}, namely D^{-1} times their full primal contribution to $\mathcal{P}_D$ in* (67)*. Then*

$$|S_{\mathrm{eq}}| \ll B_W^2R_wD^{\epsilon} \ll B_W^2R(DH)^{\epsilon}. \tag{83}$$

Before the factor D^{-1}, their ordered orientation sum at norm N is

$$\begin{aligned}a_U(N)^2|W(N/D)|^2\prod_{p|N}\{a_p\chi_p(t) + \overline{a_p\chi_p(t)}\},\\ t = Nu_0N\mathfrak{v}_1(N\mathfrak{v}_2)^2,\end{aligned} \tag{84}$$

where $a_p = \nu(\pi_p)\overline{\nu(\bar{\pi}_p)}$, $\chi_p(t) = (t/\pi_p)_6$, and $a_U(N) = \sum_{rs|N,(r,s)=1,r,s\le U}\mu_{\mathbb{Z}}(r)\mu_{\mathbb{Z}}(s)$. This includes all ordered orientations and every raw collision zero. For the all-component primitive-ideal prefix/band,

$$\frac{|\mathcal{P}_{\mathrm{eq}}(\le Y)| + |\mathcal{P}_{\mathrm{eq}}(Y,2Y]|}{D} \ll B_W^2(DYA_2)^{\epsilon}RY/D. \tag{85}$$

The signed complement is bounded by the whole primal term minus that prefix, not by an absolute high-mode summation.

Since $R \ll H$ on nonempty blocks, the desired full fixed-W target is equivalent, up to (83), to that for unequal norms. Its desired constants may depend on fixed W or appropriate specified seminorms. Formula (84) is neither identically zero nor positive in general.

Proposition 42.7 (Literal-ray recombination in the pure sector)**.** *In the pure case $\mathfrak{v}_2 = \mathcal{O}$, choose an S-supported ray modulus $\mathfrak{c} = (\gamma_c)$ divisible by (6). For an original coprime pair put $\mathfrak{q} = \mathfrak{mn}$, $Q = N\mathfrak{q}$, $\gamma_q = \alpha_{\mathfrak{m}}\alpha_{\mathfrak{n}}$, and let $\vartheta_{\mathfrak{q}}(x) = (x/\mathfrak{m})_6\overline{(x/\mathfrak{n})_6}$ be the primitive finite character modulo $\mathfrak{q}$. Let $P_{\mathfrak{k}^2}$ be the full literal primary/good-S/ray indicator on $\mathcal{O}/\mathfrak{c}$ after $v_1 = k^2 z$, with finite additive transform $\widehat{P}$. The exact common-modulus Fourier coefficient is*

$$\widehat{F}_{\mathfrak{k}^2}(\beta) = \tau_{\mathfrak{q}}\vartheta_{\mathfrak{q}}(\gamma_c)\overline{\vartheta_{\mathfrak{q}}(\beta)}\widehat{P}_{\mathfrak{k}^2}(\beta\gamma_q^{-1} \bmod \mathfrak{c}). \tag{86}$$

Its literal-lattice radial Poisson prefactor is $L_{\mathfrak{k}}/(N\mathfrak{c}Q)$, $L_{\mathfrak{k}} = A_1/(N\mathfrak{k})^2$. The S-local Fourier-mean sublattice $\beta = \gamma_c\zeta$ has coefficient $r_{\mathfrak{k}^2}\tau_{\mathfrak{q}}\overline{\vartheta_{\mathfrak{q}}(\zeta)}$, where $r = \sum P$. The six-unit projection annihilates every nontrivial unit restriction and retains the unit-trivial branch, whose condition is

$$N\mathfrak{m} \equiv N\mathfrak{n} \pmod{36}. \tag{87}$$

For the basic S above $2, 3$ with no extra ray condition, $\mathfrak{c} = (6)$ has $r = 3$ and ideal coefficient $6r/N\mathfrak{c} = 1/2$. This is not the global $\beta = 0$ mode, which vanishes. All S-supported dual ideals remain, with values defined by any generator once the character is unit-trivial.

Literal additive bands and the earlier primitive-ideal bands have different fixed-S frequency support/rescalings; no unmatched windows are identified. Unequal norms congruent modulo 36 and all other literal residue channels remain. Finite ray recombination supplies no automatic zero mean or signed target estimate.

Remaining arithmetic obligation. A finite-prefix comparison, the sparse equal-norm sector and the cubic auxiliary-band comparison do not bound the complete signed unequal-norm kernel. The required additional estimate must retain all small divisors/source classes and the literal S mean together with complementary residues before the final conductor/divisor/dual triangle. No desired positivity, energy lower bound, new zero-free strip, determinant comparison, analytic Lean or GRH claim follows.

43 An arithmetic Gauss tail with the original finite cutoff

The coherent kernel of Section 42 contains a difficult root-weighted mean. A large part of its *short-divisor expansion* can now be bounded by retaining a cubic interaction before taking a triangle. This gives a signed cutoff comparison, not a smaller norm for the original coefficients. A direct exact-discrete-frequency version is included as well. The remaining small-short-divisor moment and the separate discrete-frequency error are stated below. The whole signed power 13/12 and global unsigned power 9/8 do not change.

43.1 Original objects and a fixed common profile

Keep $K = \mathbb{Q}(\omega)$, $\mathcal{O} = \mathbb{Z}[\omega]$, $\delta = 1 + 2\omega$, and the fixed exclusions S of Section 42.1. A good ideal is prime to S. All columns are the complete good squarefree ideal annulus $D \le N\mathfrak{m} \le 2D$, $D > 1$. Equal norms do not identify ideals. For a real cutoff $1 \le T \le \sqrt{D}$ put

$$\begin{aligned} A_T(\mathfrak{m}) &= \sum_{\substack{\mathfrak{ab}\mid\mathfrak{m}\\(\mathfrak{a},\mathfrak{b})=1\\N\mathfrak{a},N\mathfrak{b}\le T}} \mu(\mathfrak{a})\mu(\mathfrak{b}),\\ c_f(\mathfrak{m}) &= f(\mathfrak{m})\nu(\mathfrak{m})R_{\mathfrak{m}}(u_0)W(N\mathfrak{m}/D). \end{aligned} \tag{88}$$

Here ν is the original fixed unitary finite-order character; u_0 is the fixed literal unit/S-supported factor with no good support. The actual coefficient is c_{A_U}, $1 \le U \le \sqrt{D}$. The function W is one fixed, possibly complex, smooth common profile supported on $[1,2]$, zero at the endpoints. Define

$$\begin{aligned} B_W &= \|W\|_\infty, \qquad \widehat{W}(t) = \int_0^\infty W(x) x^{-it}\,\frac{dx}{x}, \\ \mathcal{M}(W) &= \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{W}(t)|\,dt. \end{aligned} \tag{89}$$

This explicit seminorm is finite and controls B_W. It is a seminorm of the column profile, not of the sharp nodal row function.

Only the original unweighted *pure* two-pivot block is used: $A_2 = 1$, $\mathfrak{v}_2 = \mathcal{O}$, and the other four factors are units. The first row is good and squarefree, with the original literal/ray conditions and capacity. Its integer endpoints are

$$a = \lceil A_1 \rceil, \qquad b = \min\{\lceil 2A_1 \rceil - 1, \lfloor H/Nu_0 \rfloor\}, \qquad K = \sqrt{b}. \tag{90}$$

An empty interval contributes zero throughout. Otherwise $b \ge a \ge 1$. Use the exact G of (72), its transform $\dot{G}$ in (73), and $C_0 = 3/(4\pi^2)$. In particular $0 \le G \le 1$, $\operatorname{supp} G \subset [1/2, 5/2]$, and $\int G \le 2$. The original *pre-S* cutoff K is never recomputed after another deletion.

Let $\mathcal{I}$ be the original ordered coprime column pairs. One may also impose the *original* unequal-norm projection; the same choice is then used in every expression below. For each source pair A, B and original ray character ρ use the primitive η, $\mathfrak{f}_S$ and F of (68). Thus $F = N\mathfrak{f}_S N\mathfrak{m} N\mathfrak{n}$, and η has total infinity type zero. Put

$$\begin{aligned} T_q &= S \cup \operatorname{supp}(\mathfrak{m}\mathfrak{n}), \\ Q_s^{T_q}(K;\eta^2) &= \sum_{\substack{\mathfrak{k}\ \text{good,SF} \\ N\mathfrak{k}\le K,\ (\mathfrak{k},T_q)=1}} \frac{\mu(\mathfrak{k})\eta(\mathfrak{k})^2}{(N\mathfrak{k})^s}, \\ P_S(\eta) &= \sum_{\mathfrak{e}\mid P_S} \frac{\mu(\mathfrak{e})\eta(\mathfrak{e})}{N\mathfrak{e}}, \qquad P_S = \prod_{p\in S} p. \end{aligned} \tag{91}$$

The symbol $(\mathfrak{k}, T_q) = 1$ means avoidance of every prime in that set. Squaring η never restores a deleted prime. Values at $\mathfrak{e}$ and at any later S-supported frequency are separate primitive values, including zero.

43.2 The actual retained mean and its finite leading model

In the exact primitive kernel (74), write each dual ideal uniquely as $\mathfrak{y} = \mathfrak{h}\mathfrak{r}\mathfrak{s}^6$: $\mathfrak{h}$ is S-supported, $\mathfrak{r}$ is good and sixth-free, and $\mathfrak{s}$ is good. Overlap of $\mathfrak{r}, \mathfrak{s}$ is allowed. For $X \ge 64$ set

$$V = X^{1/6}/2, \qquad b_{\mathfrak{w}} = (X/N\mathfrak{w})^{1/6}, \qquad t_{\mathfrak{w}} = \big((N\mathfrak{w})^{-1/6} - 1/2\big)_+. \tag{92}$$

The top band is $V < N\mathfrak{s} \le 2V$, $N\mathfrak{y} \le X$. Replacing its $\mathfrak{s}$ count by its principal fixed-ray density defines the retained mean M_{top}. This operation keeps, rather than deletes, its coherent main term. The selected branches are $\rho^6 = 1$. At this top window $N\mathfrak{h}N\mathfrak{r} < 64 < 7^6$, so $\mathfrak{w} = \mathfrak{h}\mathfrak{r}$ runs over *all* integral ideals of norm below 64, including S-supported ideals.

Set

$$\kappa_S = \frac{\pi}{3\sqrt{3}} \prod_{p\in S}(1 - (Np)^{-1}), \qquad g(\mathfrak{m}) = \prod_{p\mid \mathfrak{m}}(1 - (Np)^{-1}). \tag{93}$$

On original coprime support the density is $\kappa_S g(\mathfrak{m})g(\mathfrak{n})$. For $\theta=(A,B,\rho,\mathfrak{m},\mathfrak{n},\mathfrak{e},\mathfrak{w})$ in the preceding sets, with $\rho^6=1$, define

$$\begin{aligned} d_\theta[f_1,f_2] &= \frac{\kappa_S A_1}{D} h_{AB\rho}(\mathcal{O}) c_{f_1}(\mathfrak{m})\overline{c_{f_2}(\mathfrak{n})}\frac{g(\mathfrak{m})g(\mathfrak{n})\epsilon(\eta)}{\sqrt{F}} \\ &\quad\cdot \frac{\mu(\mathfrak{e})\eta(\mathfrak{e})}{N\mathfrak{e}}\overline{\eta(\mathfrak{w})}, \qquad \lambda_\theta=\frac{A_1 N\mathfrak{w}}{N\mathfrak{e}F}. \end{aligned} \tag{94}$$

The exact definition of the normalized retained mean is

$$\begin{aligned} \mathcal{M}_{\mathrm{top}} := \frac{M_{\mathrm{top}}}{D} &= \sum_\theta d_\theta[A_U,A_U] \sum_{\substack{\mathfrak{k}\ \text{good,SF}\\ N\mathfrak{k}\le K,\ (\mathfrak{k},T_q)=1}} \frac{\mu(\mathfrak{k})\eta(\mathfrak{k})^2}{(N\mathfrak{k})^2} \\ &\quad\cdot \int_V^{b_{\mathfrak{w}}} \dot G\left(\frac{\lambda_\theta t^6}{(N\mathfrak{k})^2}\right) dt. \end{aligned} \tag{95}$$

All sums here are finite. In particular there is no $\mathfrak{k}\mid\mathfrak{w}$ or $(\mathfrak{k},\mathfrak{w})=1$ condition. The finite original source/ray coefficients $h_{AB\rho}(\mathcal{O})$ have not been centered. An argument at $\mathfrak{e}$ is not divided by a value at $\mathfrak{w}$.

Put $g_0=C_0^{-1/2}\int G$ and define

$$\mathcal{L}_0[f_1,f_2]=g_0X^{1/6}\sum_\theta d_\theta[f_1,f_2]\,t_{\mathfrak{w}}Q_2^{T_q}(K;\eta^2). \tag{96}$$

Replacing both column functions here is an explicitly defined auxiliary model, not a change to the original primal data.

Proposition 43.1 (Matched leading model with its original cutoff)**.** *For the actual coefficients and every finite $X\ge 64$,*

$$|\mathcal{M}_{\mathrm{top}}-\mathcal{L}_0[A_U,A_U]|\ll B_W^2(DX)^{\epsilon}\,\frac{A_1^2X^{7/6}}{D^2}. \tag{97}$$

The constant uses no derivative bound for the sharp G. Both interval endpoints and the original finite K occur in $\mathcal{L}_0$. This estimates the model error, not the leading signed model.

43.3 The root is an incomplete cubic Gauss weight

For each good ideal use its primary generator $\alpha_{\mathfrak{a}}\equiv 1 \pmod 3$ and write $\chi_{j,\mathfrak{a}}$ for its positive residue character of order $j=3,6$. For good squarefree $\mathfrak{m}$ put

$$\begin{aligned} G_{\mathfrak{m}} &= \frac{1}{\sqrt{N\mathfrak{m}}}\sum_{x \bmod \alpha_{\mathfrak{m}}} \chi_{3,\mathfrak{m}}(x)\exp\!\left(2\pi i\,\mathrm{Tr}\,\frac{x}{\delta\alpha_{\mathfrak{m}}}\right), \\ g_{\mathfrak{m}}(z) &= \overline{\chi_{3,\mathfrak{m}}(z)}G_{\mathfrak{m}}. \end{aligned} \tag{98}$$

The second expression is the normalized finite Gauss sum at frequency z, including zero when $(\mathfrak{m},z)>1$. For $\mathfrak{q}=\mathfrak{m}\mathfrak{n}$ put $\theta_q=\chi_{6,\mathfrak{m}}\overline{\chi_{6,\mathfrak{n}}}$. Let θ_S be the *actual finite S part* of the total type-zero η, at its own primitive conductor $\mathfrak{f}_S=(\gamma_S)$. Then

$$\mathcal{S}_{\mathfrak{m},\mathfrak{n},\rho}=\theta_S(\alpha_{\mathfrak{m}}\alpha_{\mathfrak{n}})\theta_q(\gamma_S)\frac{\tau_{\gamma_S}(\theta_S)}{\sqrt{N\mathfrak{f}_S}}. \tag{99}$$

The unit-conductor convention is one. This scalar has modulus one; it is not obtained by discarding the infinity type of a constituent. The accepted Jacobi character in (35) gives $\vartheta = \Psi/\sqrt{N}$, with conductor (2δ) of norm 12 and nonzero infinity type. The exact root formula is

$$\begin{aligned} \epsilon(\eta) &= \mathcal{S}_{\mathfrak{m},\mathfrak{n},\rho}\Gamma(\alpha_{\mathfrak{m}}\alpha_{\mathfrak{n}})\mu(\mathfrak{m})\mu(\mathfrak{n})\vartheta(\mathfrak{m})\overline{\vartheta(\mathfrak{n})}\,\overline{G_{\mathfrak{m}}}G_{\mathfrak{n}}, \\ \Gamma(a+b\omega) &= \tfrac{1}{2}\{1+i^{-b}+i^{a}+i^{b-a}\}. \end{aligned} \tag{100}$$

No extra quadratic sign is inserted. In the original kernel, $\eta(\mathfrak{k}^2)$ and the second-row phase, when present, produce the raw cubic frequency $\overline{g_{\mathfrak{m}}(\alpha_{\mathfrak{k}}\alpha_{\mathfrak{v}_2})}g_{\mathfrak{n}}(\alpha_{\mathfrak{k}}\alpha_{\mathfrak{v}_2})$; here $\mathfrak{v}_2 = \mathcal{O}$. This incomplete weighted frequency set is not a complete residue average.

43.4 A signed short-divisor tail from a physical cubic matrix

For squarefree $\mathfrak{m}$ there is an exact sign-preserving assignment identity:

$$\mu(\mathfrak{m})A_T(\mathfrak{m}) = \sum_{\substack{\mathfrak{abc}=\mathfrak{m} \\ N\mathfrak{a},N\mathfrak{b}\le T}} \mu(\mathfrak{c}). \tag{101}$$

The factors are pairwise coprime and squarefree. For $Y \le U$ the difference of cutoff boxes has two disjoint pieces. After interchanging $\mathfrak{a},\mathfrak{b}$ in the second and setting $\mathfrak{z} = \mathfrak{bc}$, the exact coefficient is

$$T_U(\mathfrak{z}) + T_Y(\mathfrak{z}), \qquad T_L(\mathfrak{z}) = \sum_{\substack{\mathfrak{b}\mid\mathfrak{z} \\ N\mathfrak{b}\le L}} \mu(\mathfrak{z}/\mathfrak{b}). \tag{102}$$

It is independent of $\mathfrak{a}$. The conjugate Gauss CRT identity retains the cross interaction

$$\overline{G_{\mathfrak{az}}} = \overline{G_{\mathfrak{a}}G_{\mathfrak{z}}}\chi_{3,\mathfrak{a}}(\alpha_{\mathfrak{z}}), \qquad (\mathfrak{a},\mathfrak{z}) = 1. \tag{103}$$

The raw cubic entry is zero at a common prime.

The deep input is explicitly external: de Faveri, arXiv:2610.04045v1, Theorem 1.1 and Example 2.2, at $n = 3$. For the physical primary cubic matrix on good squarefree ideals of norms comparable to A, Z it gives

$$\|[\chi_{3,\mathfrak{a}}(\alpha_{\mathfrak{z}})]\|_{\mathrm{op}}^2 \ll_\epsilon (AZ)^\epsilon\{A + Z + (AZ)^{2/3}\}. \tag{104}$$

Both indices are actual ideals and hence cube-free, not arbitrary primitive-conductor labels. This is the same pinned input as the cubic comparison in Proposition 42.5.

Theorem 43.2 (Actual Gauss-weighted cutoff tail)**.** *For any bounded multiplicative ideal function λ on good ideals, with $|\lambda| \le 1$, and $1 \le Y \le U \le \sqrt{D}$,*

$$\begin{aligned} \left|\sum_{\mathfrak{m}\ \mathrm{good,SF}} \frac{\mu(\mathfrak{m})(A_U - A_Y)(\mathfrak{m})\overline{G_{\mathfrak{m}}}\lambda(\mathfrak{m})}{\sqrt{N\mathfrak{m}}} W(N\mathfrak{m}/D)\right| \\ \ll_\epsilon \mathcal{M}(W)D^\epsilon\{D^{1/3} + \sqrt{D/Y}\}. \end{aligned} \tag{105}$$

The constant is uniform in the conductor or raw deletion set in λ. For the actual pure leading model, with every original finite K, class/ray/root/S component and coprime pair retained,

$$|\mathcal{L}_0[A_U, A_U] - \mathcal{L}_0[A_Y, A_Y]| \ll B_W\mathcal{M}(W)D^\epsilon A_1 X^{1/6}\{Y^{-1/2} + D^{-1/6}\}. \tag{106}$$

The original unequal-norm projection is also allowed.

This uses the cubic interaction before the short-divisor triangle. It does not assert pointwise or coefficient-norm shrinkage of A_U, nor a separate cancellation theorem for $\mu(\mathfrak{c})$.

43.5 What has been reduced, and what remains

Combining Proposition 43.1 and Theorem 43.2 gives

$$|\mathcal{M}_{\mathrm{top}} - \mathcal{L}_0[A_Y, A_Y]| \ll (DX)^{\epsilon} \Big\{ B_W^2 \frac{A_1^2 X^{7/6}}{D^2} + B_W \mathcal{M}(W) A_1 X^{1/6} (Y^{-1/2} + D^{-1/6}) \Big\}. \tag{107}$$

At $A_1 \asymp H = D^{3/4}$, $X = D^{3/8}$ and $Y = \min\{U, D^{1/8}\}$, the errors are target-sized. If $Y = U$ the coefficient tail is exactly zero; otherwise the three displayed powers are $-1/16$, $3/4$ and $31/48$. The remaining exact coefficients satisfy

$$\mu(\mathfrak{m}) A_Y(\mathfrak{m}) = \sum_{\substack{\mathfrak{abc}=\mathfrak{m} \\ N\mathfrak{a}, N\mathfrak{b} \le Y}} \mu(\mathfrak{c}), \qquad N\mathfrak{c} \ge D/Y^2 \ge D^{3/4}. \tag{108}$$

The small-short-divisor, long residual-Möbius cubic Gauss bilinear moment $\mathcal{L}_0[A_Y, A_Y]$ is still unestimated at the target.

43.6 The exact discrete top prefix, with every ray retained

There is also a direct reduction that does not introduce the principal-density split. Its exact support is

$$\mathcal{T}_X = \{(\mathfrak{w}, \mathfrak{s}) : N\mathfrak{w} < 64,\ \mathfrak{s} \text{ good},\ V < N\mathfrak{s} \le 2V,\ N\mathfrak{w}(N\mathfrak{s})^6 \le X\}. \tag{109}$$

There is no squarefree restriction on $\mathfrak{s}$. Overlap of $\mathfrak{w}, \mathfrak{s}$ and of $\mathfrak{k}, \mathfrak{s}$ remains allowed. Let $\widetilde{\Theta}$ be the same outer set as θ above, but with *every* original ρ, and put

$$\widetilde{d}_\theta[f_1, f_2] = \frac{A_1}{D} h_{AB\rho}(\mathcal{O}) c_{f_1}(\mathfrak{m}) \overline{c_{f_2}(\mathfrak{n})} \frac{\epsilon(\eta)}{\sqrt{F}} \cdot \frac{\mu(\mathfrak{e})\eta(\mathfrak{e})}{N\mathfrak{e}} \overline{\eta(\mathfrak{w})}. \tag{110}$$

This is an explicit definition, not division by a density or a primitive value. With the original $\mathfrak{k}$ support as in (91), define the normalized exact piece

$$\mathcal{P}_X[f_1, f_2] = \sum_{\theta \in \widetilde{\Theta}} \widetilde{d}_\theta[f_1, f_2] \sum_{\mathfrak{k}} \frac{\mu(\mathfrak{k})\eta(\mathfrak{k})^2}{(N\mathfrak{k})^2} \cdot \sum_{\mathfrak{s}:\ (\mathfrak{w}, \mathfrak{s}) \in \mathcal{T}_X} \bar{\rho}(\mathfrak{s})^6 \mathbf{1}_{(\mathfrak{s}, \mathfrak{mn})=1} \dot{G}\left(\frac{\lambda_\theta(N\mathfrak{s})^6}{(N\mathfrak{k})^2} \right). \tag{111}$$

Then $\mathcal{P}_X[A_U, A_U]$ is the actual original top primitive-prefix contribution divided by D. It is a signed representation-fixed piece, not a positive energy or the whole infinite dual sum.

Let $\mathcal{L}_X^K[f_1, f_2]$ be exactly (111) with $\dot{G}$ replaced by g_0. Only in that leading model can the finite $\mathfrak{k}$ sum be factored as $Q_2^{T_q}(K; \eta^2)$. The discrete $\mathfrak{s}$ sum and its endpoints remain. Unlike $\mathcal{L}_0$, this model has no $\kappa_S g(\mathfrak{m}) g(\mathfrak{n})$ density, no $\rho^6 = 1$ projection and no integral replacing the $\mathfrak{s}$ family.

Proposition 43.3 (Direct original-prefix cutoff reduction)**.** *Put $\Lambda = A_1 X / D^2$. With precisely the original support,*

$$|\mathcal{P}_X[A_U, A_U] - \mathcal{L}_X^K[A_U, A_U]| \ll B_W^2 (DX)^{\epsilon} A_1 V \Lambda, \tag{112}$$

and

$$|\mathcal{L}_X^K[A_U, A_U] - \mathcal{L}_X^K[A_Y, A_Y]| \ll B_W \mathcal{M}(W)(DX)^\epsilon A_1 V \cdot \{Y^{-1/2} + D^{-1/6}\}. \tag{113}$$

In particular the actual original piece satisfies

$$|\mathcal{P}_X[A_U, A_U] - \mathcal{L}_X^K[A_Y, A_Y]| \ll (DX)^\epsilon A_1 V\{B_W^2 \Lambda + B_W \mathcal{M}(W)(Y^{-1/2} + D^{-1/6})\}. \tag{114}$$

The original unequal-norm projection is allowed.

At the critical pure parameters used above the three powers are again $-1/16$, $3/4$ and $31/48$, with exact zero coefficient tail when $Y = U$. This is a reduction of the *actual discrete* piece to a different, still unestimated, small-short-divisor model. That model retains every ρ, every nonunit S mode, the exact $\mathfrak{w}, \mathfrak{s}$ support, root/conductor data, original finite K and both long residual-Möbius column coefficients.

Separate open errors and scope. The accepted fixed-ray disk count gives the distinct top-band certificate

$$\frac{|\mathcal{P}_{\text{top}} - M_{\text{top}}|}{D} \ll B_W^2 (DX)^\epsilon A_1 X^{1/18}\{1 + \sqrt{A_1 X/D^2}\}. \tag{115}$$

Its pure critical power is $37/48 > 3/4$; this is an upper certificate, not a lower bound for the actual error. The direct reduction in Proposition 43.3 does not introduce that split: its discrete model retains those effects. No improved bound for the separate discrepancy, for the remaining small-cutoff models, or for the other original sectors is proved here. The result is an application of a named external cubic sieve to an actual arithmetic tail, not a new sieve, a complete analytic Lean formalization, a determinant/zero-location bridge or a proof of GRH. Fixed-profile constants and the explicit $\mathcal{M}(W)$ cannot be replaced by a uniform supnorm-only claim for varying profiles.

44 Ramified theta transforms and the original pure dual prefix

The exact discrete prefix introduced in Section 43 can be estimated with its original coefficient A_U. The argument constructs the previously missing finite-character transform at the ramified prime and groups all original short and Heath–Brown factors before taking a short-factor triangle. The resulting power is $101/144$ for this specified pure finite dual prefix. The whole signed comparison $13/12$, the global unsigned comparison $9/8$, and the remaining infinite-frequency assembly retain their earlier status.

The retained principal-density mean in the earlier discussion is a different expression with its own comparison problem. The result here uses every original ray component and the actual discrete sixth-part set. It supplies a new proof for the actual finite discrete prefix rather than modifying the historical cutoff stages.

44.1 The normalized object and its original support

Put $\omega = e^{2\pi i/3}$, $K = \mathbb{Q}(\omega)$, $\mathcal{O} = \mathbb{Z}[\omega]$ and $\lambda = 1 + 2\omega = i\sqrt{3}$. A good ideal avoids the original fixed set S, containing the primes above $2, 3$. All ideal columns are good and squarefree; different ideals of equal norm remain different columns. Use the unique multiplicative primary generator

$\alpha_{\mathfrak{a}} \equiv 1 \pmod 3$. The original character ν has fixed finite order, with its conductor exclusions in S; no order-dividing-six hypothesis is imposed on ν.

Assume

$$D \ge 2^{64}, \quad 1 \le U \le \sqrt{D}, \quad A_1 \asymp H = D^{3/4}, \quad X = D^{3/8}, \quad V = X^{1/6}/2. \tag{116}$$

Only the original unweighted pure two-pivot block is asserted: $A_2 = 1$, its second good factor and the other four good factors are units, and u_0 is the fixed literal unit/S-supported factor with no good prime support. Keep its original capacity endpoints, nodal function G, and pre-S cutoff $K_{\mathrm{cut}} = \sqrt{b}$. An empty capacity interval gives zero. None of these data is recomputed during a coefficient decomposition.

Let W be one fixed possibly complex smooth function supported on $[1,2]$, with $W(1) = W(2) = 0$, and write

$$\begin{aligned} B_W &= \|W\|_\infty, \qquad \widehat{W}(t) = \int_0^\infty W(x) x^{-it} \frac{dx}{x}, \\ \mathcal{M}(W) &= \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{W}(t)|\, dt, \qquad\qquad V_1(W) = B_W + \int_1^2 |W'(x)|\, dx. \end{aligned} \tag{117}$$

These are seminorms of the common column profile. No uniform derivative hypothesis on the sharp capacity interpolant is added. The exact frequency set is

$$\mathcal{T}_X = \left\{ (\mathfrak{w}, \mathfrak{s}) : \begin{array}{l} \mathfrak{w} \text{ integral, } N\mathfrak{w} < 64, \quad \mathfrak{s} \text{ good integral,} \\ V < N\mathfrak{s} \le 2V, \quad N\mathfrak{w}(N\mathfrak{s})^6 \le X \end{array} \right\}. \tag{118}$$

There is no squarefreeness requirement on $\mathfrak{s}$. The permitted overlaps among $\mathfrak{k}, \mathfrak{s}, \mathfrak{w}$ remain permitted, including nonunit S-supported $\mathfrak{w}$.

Here is a full definition of the piece being bounded. Let $\Omega = (A, B, \rho, \mathfrak{m}, \mathfrak{n})$ range over the original finite source/ray components and their ordered coprime good squarefree columns in $[D, 2D]$, with the original source classes. An original unequal-norm projection may be imposed consistently. Use the actual primitive character η_Ω of (68), its actual S-conductor $\mathfrak{f}_S$, and

$$F_\Omega = N\mathfrak{f}_S\, N\mathfrak{m}\, N\mathfrak{n}, \qquad \mathfrak{q} = \mathfrak{m}\mathfrak{n}, \qquad c_U(\mathfrak{m}) = A_U(\mathfrak{m}) \nu(\mathfrak{m}) R_{\mathfrak{m}}(u_0) W(N\mathfrak{m}/D).$$

The numerator value $R_{\mathfrak{m}}(u_0)$ includes its original raw zeros. Put $P_S = \prod_{\mathfrak{p} \in S} \mathfrak{p}$. With $C_0 = 3/(4\pi^2)$, the normalized prefix is

$$\begin{aligned} \mathcal{P}_X[A_U, A_U] = \frac{A_1}{D} \sum_{\Omega} h_{AB\rho}(\mathcal{O}) c_U(\mathfrak{m}) \overline{c_U(\mathfrak{n})} \frac{\varepsilon(\eta_\Omega)}{\sqrt{F_\Omega}} \\ \cdot \sum_{\mathfrak{e} \mid P_S} \frac{\mu(\mathfrak{e}) \eta_\Omega(\mathfrak{e})}{N\mathfrak{e}} \sum_{\substack{\mathfrak{k} \text{ good,SF} \\ N\mathfrak{k} \le K_{\mathrm{cut}}}} \frac{\mu(\mathfrak{k}) \eta_\Omega(\mathfrak{k})^2}{(N\mathfrak{k})^2} \\ \cdot \sum_{(\mathfrak{w}, \mathfrak{s}) \in \mathcal{T}_X} \overline{\eta_\Omega(\mathfrak{w})}\, \overline{\rho(\mathfrak{s})}^6 \mathbf{1}_{(\mathfrak{s}, \mathfrak{q}) = 1} \dot{G}\left(\frac{A_1 N\mathfrak{w} (N\mathfrak{s})^6}{(N\mathfrak{k})^2 N\mathfrak{e} F_\Omega} \right). \end{aligned} \tag{119}$$

All sums are finite. The ray label ρ here is distinguished from the Jacobi column weight below. Values at $\mathfrak{e}$ and $\mathfrak{w}$ are separate primitive values, including zero; a conductor drop does not restore a deleted prime. The original $1/D$ occurs once. The raw numerator is $D\mathcal{P}_X$.

Define $\mathcal{L}_X^{K_{\mathrm{cut}}}$ by replacing $\dot{G}$ in (119) by $g_0 = C_0^{-1/2} \int_0^\infty G(v)\, dv$, with every other datum fixed. For the original coefficients the established real-axis estimate is

$$|\mathcal{P}_X - \mathcal{L}_X^{K_{\mathrm{cut}}}| \ll B_W^2 (DX)^{\epsilon} A_1 V \Lambda, \qquad \Lambda = A_1 X / D^2. \tag{120}$$

It uses $0 \le G \le 1$, its fixed compact support and a pointwise Bessel remainder. The value g_0 may vanish and is never divided out.

44.2 Actual finite characters and a sharp ramified row

The actual root calculation leaves one angular Jacobi gauge and a cubic Gauss root in each varying column. On good squarefree ideals the Jacobi function is $\rho_J = \mu\vartheta$, where $\vartheta = \Psi/\sqrt{N}$ has infinity type $(1,0)$; here $\Psi(\mathfrak{a})$ is the chosen generator 1 mod 2λ. The arithmetic μ belongs in the coefficient opening. There is exactly one varying ϑ, or its conjugate; an additional finite-order assertion about ρ_J would be false.

Write $\chi_{3,\mathfrak{a}}$ for the positive cubic residue character, with its zero at every shared prime. The source Gauss root is

$$G_{\mathfrak{a}}^{\mathrm{src}} = \frac{1}{\sqrt{N\mathfrak{a}}} \sum_{x \bmod \alpha_{\mathfrak{a}}} \chi_{3,\mathfrak{a}}(x) e^{2\pi i \operatorname{Tr}(x/\alpha_{\mathfrak{a}})}. \tag{121}$$

The actual root uses $\lambda\alpha_{\mathfrak{a}}$ in this additive denominator. The normalization of the angular factor is explicit:

$$\begin{aligned} \Theta_\ell(\mathfrak{a}) &= (\alpha_{\mathfrak{a}}/|\alpha_{\mathfrak{a}}|)^\ell, \\ \vartheta(\mathfrak{a}) &= \Theta_1(\mathfrak{a})\overline{\chi_{3,\mathfrak{a}}(2)}, \\ G_{\mathfrak{a}}^{\mathrm{act}} &= \chi_{3,\mathfrak{a}}(\lambda) G_{\mathfrak{a}}^{\mathrm{src}}. \end{aligned} \tag{122}$$

Thus the two fixed corrections may be written as source frequency $12t$. The factor 12 belongs to the fixed finite character data; $12t$ is not declared a good squarefree moving level.

After exact product-character expansion of the joint finite S/source/Γ selectors, every remaining own-variable phase has the form $\Theta_{\pm 1}\kappa$, where κ belongs to a fixed finite residue family, with its original modulus-zero extension. It includes higher 3-primary conductors of ν. The finite source quotient is a fixed ray quotient; fixed numerator symbols are finite Kummer characters. The full Fourier sums retain their actual coefficient masses and reconstruct the primitive $\mathfrak{e}/\mathfrak{w}$ zeros. Growing other-column deletions remain literal masks.

For a physical good squarefree frequency $\mathfrak{t}$, an exact interval $I \subset [T, 2T]$, and a finite good deletion ideal Q, write the actual row in either required conjugate orientation as

$$\mathcal{F}_\psi(T; \mathfrak{t}; Q, I) = \sum_{\substack{\mathfrak{a} \text{ good,SF} \\ N\mathfrak{a} \in I,\ (\mathfrak{a},Q)=1}} \psi(\mathfrak{a}) \overline{G_{\mathfrak{a}}^{\mathrm{act}}} \chi_{3,\mathfrak{a}}(\alpha_{\mathfrak{t}}) \frac{W(N\mathfrak{a}/T)}{\sqrt{N\mathfrak{a}}}. \tag{123}$$

Here ψ is from this actual fixed family, including its angular factor. It is not an arbitrary bounded multiplicative function.

Theorem 44.1 (The actual fixed-family sharp row)**.** *For every such member, both conjugate orientations and every physical good squarefree* $\mathfrak{t}$*, including the unit,*

$$|\mathcal{F}_\psi(T; \mathfrak{t}; 1, I)| \ll_{\mathrm{fixed},\epsilon} V_1(W)(2 + TN\mathfrak{t})^\epsilon \min\{\sqrt{T},\ T^{1/4}(N\mathfrak{t})^{3/8}\}, \qquad T \ge 1. \tag{124}$$

The constant may depend on the fixed conductor/character family. A literal unit calculation extends the estimate to $1/2 \le T < 1$*; for* $T \le 1/2$ *the row is zero with the stated endpoint convention.*

Proof. Use the unconditional theta modularity and known 27 integral cusps in Dunn–Radziwiłł, arXiv:2109.07463v3, Section 5 and Appendix A. Its weight-zero cubic theta θ is automorphic with Kubota multiplier χ on $\Gamma_2 = \mathrm{SL}_2(\mathbb{Z})\Gamma_1(3)$, where $\Gamma_1(3)$ is the principal congruence subgroup of $\mathrm{SL}_2(\mathcal{O})$. Fix its known representatives γ_j and $F_j = \theta \circ \gamma_j$. The correct ramified coefficient projector

acts on $\lambda^3\mu$. Its primary selector is $\mathbf{1}_{\lambda^3\mu\equiv 1 \bmod 3}$. A finite Fourier expansion writes it, and the away-3 periodic twist, as actual rational translates of theta.

For $a = A/C$ reduced, choose $\gamma = \left(\begin{smallmatrix} A & B \\ C & D_0 \end{smallmatrix}\right)$ of determinant one, and choose j so that $\delta = \gamma\gamma_j^{-1} \in \Gamma_2$. Use $U(x) = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ and $W_C = \left(\begin{smallmatrix} 0 & -1/C \\ C & 0 \end{smallmatrix}\right)$. Then

$$\gamma^{-1}U(a) = U(-D_0/C)W_{-C}, \qquad \theta(U(a)w) = \chi(\delta)F_j(U(-D_0/C)W_{-C}w).$$

The multiplier is its literal Kubota value. Pure nonzero angular derivatives kill the pulled-back radial constants. Their Mellin identity has factor $(NC)^{1-2s}$, including the Jacobian. The explicit cusp coefficients have absolute Dirichlet norm at most

$$3^8\zeta_K^{(\lambda)}(\sigma)\zeta_K^{(\lambda)}(3\sigma - \tfrac{1}{2}), \qquad 1 < \sigma < 13/12.$$

Normalized finite Fourier mass is $O_{\text{fixed}}(\sqrt{N\mathfrak{t}})$ and each reduced denominator has norm $O_{\text{fixed}}(N\mathfrak{t})$. This gives the conservative dual cost $(N\mathfrak{t})^{3/2+\epsilon}$.

Coefficientwise cube completion retains the raw $L(3s - \frac{1}{2}, \kappa^3\Theta_{3\ell})$, with every stipulated Euler omission. Apply convexity to this entire completed numerator first. Its reciprocal is used only where $\Re(3s - \frac{1}{2}) > 1$, by absolutely convergent Euler products. Exact half-integer Perron thresholds and partial summation give (124). The optimized height is $(T/(N\mathfrak{t})^{3/2})^{1/4}$ when this is at least one; the elementary count gives the other branch. The extended edition retains the constants and the zeroth-angular residues explicitly. □

44.3 The full original coefficient and its finite prefix

On the actual contributing annulus $N\mathfrak{m} > U$, put

$$B_U(\mathfrak{m}) := \mu(\mathfrak{m})A_U(\mathfrak{m}) = \sum_{\substack{\mathfrak{abc}=\mathfrak{m} \\ N\mathfrak{a},N\mathfrak{b}\le U}} \mu(\mathfrak{c}). \tag{125}$$

This B_U is not the project's original C_U. Equation (125) is not asserted for arbitrary small columns or the unit. Unit factors inside an annular factorization remain present.

Theorem 44.2 (Original pure finite dual-prefix bound)**.** *Under* (116)*, with the complete original support and fixed data of* (119)*,*

$$\begin{aligned} |\mathcal{P}_X[A_U, A_U]| \ll_{\text{fixed},\epsilon} (DXA_1)^{\epsilon}A_1V\{&B_W[\mathcal{M}(W) + V_1(W)]D^{-1/9} \\ &+ B_W^2[\Lambda + D^{-1}]\}. \end{aligned} \tag{126}$$

In particular its normalized leading power is

$$13/16 - 1/9 = 101/144 < 3/4, \qquad 3/4 - 101/144 = 7/144. \tag{127}$$

The raw numerator has one additional factor D.

Proof. Open (125) and the exact $J = 5$ finite Heath–Brown identity at $z_0 = (2D)^{1/5}$. There are at most eleven factors, including the original $\mathfrak{a}, \mathfrak{b}$. Choose their grouping from fixed independent dyadic scales. A balanced whole-factor partition has a physical cubic matrix bound $\mathcal{M}(W)D^{1/3+\epsilon}$, using the pinned de Faveri, arXiv:2610.04045v1, Theorem 1.1 and Example 2.2, at order three.

An unbalanced box has one long unrestricted coefficient-one factor $\mathfrak{v}$. At $D \ge 2^{64}$ neither an original short factor nor a short Möbius factor can occupy that scale. The product $\mathfrak{p}$ of all other

factors has norm $O(D^{1/3})$ and its intrinsic signed assignment coefficient is divisor-bounded. This retains the $j = 1$ boxes, which are all balanced.

For $N\mathfrak{p} > Z$ the cubic matrix gives $\mathcal{M}(W)D^{\epsilon}[D^{1/3} + \sqrt{D/Z}]$. For $N\mathfrak{p} \le Z$, freeze the original opposite column $\mathfrak{n}$ first. Keep $\mathbf{1}_{(\mathfrak{p},\mathfrak{n})=1}$ and all prefix $\mathfrak{k}/\mathfrak{s}/\mathfrak{w}$ zeros in its actual coefficient. The remaining long-variable deletion is $Q = \mathrm{rad}_{\mathrm{good}}(\mathfrak{p}\mathfrak{n}\mathfrak{k}\mathfrak{s}\mathfrak{w})$. Its exact divisor $\mathfrak{h}$ changes frequency to $\mathfrak{p}\mathfrak{k}\mathfrak{h}$ and length to $D/(N\mathfrak{p}N\mathfrak{h})$. Including the original prefix weight, Theorem 44.1 gives

$$\frac{1}{\sqrt{N\mathfrak{p}N\mathfrak{h}}}\left(\frac{D}{N\mathfrak{p}N\mathfrak{h}}\right)^{1/4}(N(\mathfrak{p}\mathfrak{k}\mathfrak{h}))^{3/8} = D^{1/4}(N\mathfrak{k})^{3/8}(N\mathfrak{p})^{-3/8}(N\mathfrak{h})^{-3/8}. \tag{128}$$

Thus the small-prefix column costs $V_1(W)D^{\epsilon}D^{1/4}(N\mathfrak{k})^{3/8+\epsilon}Z^{5/8}$. Choose $Z = D^{2/9}$; both column costs have power $7/18$.

Only after that bound take the opposite-column elementary $B_W D^{1/2+\epsilon}$ count. Use the original A_1/D once, sum the original finite $\mathfrak{k}$ with the convergent positive majorant $\sum(N\mathfrak{k})^{-13/8+\epsilon}$, and count the exact $\mathfrak{s}$ family by $O(V)$. Optional unequal norms cost the same fixed-norm fibre bound $O(B_W^2 A_1 V D^{-1+\epsilon})$. Finally (120) applies directly to A_U, A_U. No A_U-to-A_Y replacement or common-long-divisor opening is used. □

The estimate has fixed-conductor and explicit profile-seminorm dependence. It does not supply a supremum-norm estimate for varying W_D, an arbitrary weighted-row operator, the infinite dual complement, mixed or neighboring blocks, short/principal assembly, an independent positive determinant, or a zero-location theorem. The Dirichlet, Hecke, Dedekind, elliptic and general Artin obligations remain separate. No analytic/category Lean formalization or GRH proof is claimed.

45 Paired cutoffs, exact divisor signs and proved regimes

The comparison in this section concerns a specified part of the signed coprime-column form for the original unweighted, nonprincipal, two-pivot Eisenstein rows. We use the physical ideal norms and the original ordered cutoff. Let D, U, H be the column, cutoff and capacity parameters, with $U^2 \le D \le 7U^2/2$ and $A_1A_2^2 \ll H \le D$. The first two row ideals have dyadic norms A_1, A_2, $R = A_1A_2$, and the actual primitive conductor of row u has norm $Q_u \asymp R$. The finite bad-prime set S, row type, and twist are fixed. The same smooth complex profile W supported on $[1, 2]$ is used on both sides. Ideal labels, including conjugate primes of the same norm, are retained individually.

45.1 The selected pairing

For each good squarefree intermediate Möbius divisor d, put $E = Nd$. Let V_d be its original row amplitude and $\mathcal{E}_d$ the equal-norm part of the original middle Poisson window. We study selections of

$$\frac{2}{D}\,\mathrm{Re}\sum_d \mu(d)\langle \mathcal{E}_d, V_d - \mathcal{E}_d\rangle. \tag{129}$$

Here the counting inner product is linear in the first argument. This is one term in the expansion of the original signed norm. In particular the intermediate d is different from the gcd of the final column pair.

On an original allocation let $z = e_0z_0$ with $J \le Nz < 2J$, let $e_1 \mid \mathrm{rad}(Sd)$ be the other deletion label, and put $h = re_0$, with r the original squarefree short divisor. Set

$$X = \frac{D}{E(Nr)^2}, \qquad X' = \frac{X}{(Ne_0)^2}, \qquad M = \frac{R\,Ne_0\,Ne_1\,J}{X}.$$

The original middle window has $X' > B_*$ and $K_0 < Nk \le MD^\eta$, for its specified B_*, K_0, η. The primal test discards the guaranteed zero cells $X/((Ne_0)(Ne_1)J) \le 1/2$, and the original weights retain any further zeros. An allocation with $X' \le B_*$ belongs to the direct component. Its re-expressed Poisson sum stays in the complement in (129).

On the equality side write $z_{0,0} = g\ell$ and $k_0 = g\bar{\ell}$, so $Nz_{0,0} = Nk_0$. On the partner side the selected changes in the good exponents consist of erasing the squarefree cutoff part x, erasing a part of the good conjugate overlap, and inserting allowed positive increments at other good primes outside d. A conjugate erasure removes the full original exponent, which can be two. At an insertion prime the old residue plus the increment is at most five. Changes at the good primes of d keep their exact new residues. For fixed other labels write $z_{0,1} = z_*x$ and $k = Ik_*$. The physical insertion ideal I keeps each exponent $1, \dots, 5$ and is disjoint from the selected erasure supports and d. Thus $Nx \asymp B$ and $NI \asymp K$ are physical lengths. The two varying raw phases are $\psi_u(x)\overline{\psi_u(I)}$, or their conjugates if the pairing is conjugated. The fixed row factors retain the union of deleted primes, the literal section, the twist and the primitive S-values.

The subtraction on the partner imposes $Nz_*Nx = Nk_*NI$ only in its original middle window. At fixed x this fixes at most one integer norm for I, and the reverse count fixes at most one norm for x. An integer norm supports at most divisor-many ideals on the polynomial clip. With the physical inverse $1/Nx$, the paired count therefore costs $\min(1, K/B)$ up to subpowers. The two half radial powers leave

$$(B/K)^{1/2} \min(1, K/B) = \sqrt{\min(B/K, K/B)}.$$

The remaining harmonic weights give $O(R/E)$ for this subtraction at each d, with the inherited profile and subpower costs. Both original endpoints remain in the count. Equal norms in the direct and other components remain in the pairing.

45.2 Target bounds on the selected cross

The bounds have the common cost $O_{\epsilon,S,\nu,u_0}((DH)^\epsilon \mathcal{M}_{24}(W)^2)$, where $\mathcal{M}_{24}$ controls the indicated smooth profile derivatives. The live partner primal test supplies $E \ll D/B$ and partner radial scale $y_1' \gg Nk/R \gg K/R$. The proof first sums each actual column array. It then uses the row comparisons and the original label sums.

At the mixed scales $A_1 \asymp D^{5/12}$, $A_2 \asymp D^{1/6}$, the selected normalized cross has upper power $57/80$ over all physical ratios and all d. If $H \ll D^{3/4}$, a perturbed primitive sharp moment gives the target power $3/4$ for the same selected cross on

$$\frac{7}{24} \le \log_D A_1 \le \frac{439}{912}, \qquad A_1 A_2^2 \ll H.$$

The second pivot need only satisfy the stated original admissibility and capacity. The conductor comparison uses the actual R.

At the pure critical scales $A_1 \asymp R \asymp D^{3/4}$, $A_2 \asymp 1$, the target power $3/4$ holds over all d in each of the following physical sectors:

- $K \gtrsim D^{1089/1420}$, with any erasure length B.
- $B \gtrsim D^{1355/1482}$, with any insertion length K.
- $B \gtrsim R$ and $K \lesssim D^{921/1652}$.

The flexible central windows give further short and long threshold curves. There are also separate comparisons for a small fixed residual label product and for a small d-prefix. The comparable pure

all-divisor comparison has power 55/68, and the target holds on $Nd \le D^{9/68}$ in that sector. Each assertion concerns the selected changes described above. The remaining pure rectangles and the coherently averaged signed divisor sum are open.

45.3 Ordered assignment signs

Fix an ordered factorization $\zeta = \zeta_1\zeta_2$ of the actual squarefree cutoff label $\zeta = e_0z_0$ on either side, with $(d,\zeta) = 1$ and $N\zeta_i \le U$. Write $T_i = U/N\zeta_i$. The assignment sum multiplying $\mu(\zeta)$ is

$$F_d = \sum_{\substack{a,b\mid d,\ (a,b)=1\\ Na\le T_1,\ Nb\le T_2}} \mu(a)\mu(b).$$

Complete the b-sum and then subtract its excluded terms. With $m = d/b$ in the subtraction, squarefreeness gives the exact identity

$$\frac{F_d}{\mu(d)} = \mathbf{1}_{E\le T_1} + \mathbf{1}_{E\le T_2} - 1 - \sum_{\substack{m\mid d\\ T_1<Nm<E/T_2}} \sum_{\substack{a\mid m\\ Na\le T_1}} \mu(m/a). \tag{130}$$

The complete b-sum forces $a = d$. The unit $m = 1$ gives the indicator at T_2, while the nonunit terms with $Nm \le T_1$ vanish by the full inner Möbius sum. Both inequalities of the remaining band are strict. Below the product cap the band is empty. The three possible assignment values are $\mu(d), 0, -\mu(d)$ according as both, just one, or neither individual capacity reaches E.

Fix the row, both ordered factorizations and the primal and dual labels. Select both e_1 labels to be fixed S-supported ideals, take the good changed-residue ideal on d to be the unit, and require d to avoid the full fixed prime union of the row and the primal and dual labels. When $EN\zeta \le U^2$ on both sides, their capacity step functions multiply the original outer Möbius sign. The fixed-field deleted Möbius estimate supplies subpower cancellation for the resulting fixed-label scalar on $Nd \asymp Y$ with $Y \ge D^\kappa$ for a fixed $\kappa > 0$. In the pure comparable block, a physical Nx-interval of width $O(BD^{-1/8})$ gives the target after the full divisor sum by a separate sparse erased-column estimate. This includes the narrow interval immediately above the product cap on the actual partner allocation. The wide correction and the remaining coherent signed estimate require additional arguments. The whole signed and unsigned powers 13/12 and 9/8, the independent determinant comparison, and the scope-specific zero-location obligations remain as in the preceding sections.

46 Wide cutoff corrections and full-frequency compensation

Keep the pairing (129), the exact ordered cutoff identity (130), and all the original allocation, primitive-conductor and raw-zero conditions. At one live allocation put $h = re_0$, $E_1 = Ne_1$, and write the actual cutoff ideal as $\zeta = e_0z_0 = \zeta_1\zeta_2$. Thus $T_i = U/N\zeta_i \ge 1$, $E = Nd$, and

$$\Omega = \frac{E}{T_1T_2} = \frac{EN\zeta}{U^2} < \frac{14}{E_1Nh}.$$

Let $\mathcal{P}$ consist of the good prime ideals of norms seven or thirteen, with the original S-deletion. These are at most four separate ideal labels. Put $s(E) = \mathbf{1}_{E\le T_1} + \mathbf{1}_{E\le T_2} - 1$. The exact wide correction is

$$\begin{aligned} F_d(T_1,T_2) &= \mu(d)\big(s(E) + C_d(T_1,T_2)\big),\\ C_d(T_1,T_2) &= \sum_{\substack{mn=d\\ Nm>T_1,\ Nn>T_2}} \left(1 - \sum_{\substack{p\in\mathcal{P}\\ p\mid m}} \mathbf{1}_{Nm>T_1Np}\right). \end{aligned} \tag{131}$$

The two factors are coprime because d is squarefree. The strict tests retain the original cutoff endpoints. For $E_1Nh \geq 2$ only the unweighted factorization count remains. For $E_1Nh \geq 14$ the correction vanishes.

At fixed row and fixed primal and dual labels, prescribe the exact intersection δ of d with the full forbidden prime union, including $\mathcal{P}$. In a divisor dyad $Y \leq Nd < 2Y$ put $Y' = Y/N\delta$. For fixed $\kappa > 0$ and $Y' \geq D^{\kappa}$, the fixed-label scalar after taking out its fixed coefficient and row phase has bound

$$\ll_{\kappa} \mathcal{M}_{24}(W)^2 \frac{4^{\omega(\delta)} \log^C(2D)}{Y\, N\delta} \exp(-c_{\kappa}\sqrt{\log(2Y')}). \tag{132}$$

The estimate includes both wide corrections, with the original full prime deletions. It gives subpower cancellation before a coherent average over the rows and other labels.

For each $p \in \mathcal{P}$, replacing the partner coefficient by its signed p-subtraction in (131) gives a zero primal amplitude $V_d^{[p]}$. The exact full-frequency complement is $V_d^{[p]} - \mathcal{E}_d^{[p]} = -\mathcal{E}_d^{[p]}$, where $\mathcal{E}_d^{[p]}$ is its middle equality projection. The original elementary equality norm gives a contribution of order at most $R(DH)^{\epsilon}\mathcal{M}_{24}(W)^2$ after the full divisor sum. This comparison requires all partner modes, including the re-expressed direct component.

In the pure comparable sector $A_1 \asymp R \asymp D^{3/4}$, $A_2 \asymp 1$, $B, K \asymp R$, the equality-side wide correction reaches $D^{3/4+\epsilon}$ on $Nd \leq D^{13/56}$ with the inherited profile cost. Its partner can carry the original coefficient, the step part, or the correction part. Thus this prefix includes their double correction. The remaining coherent wide sum and the other signed components remain open.

47 Joint moments for the wide cutoff pairing

We work at the pure scales $A = A_1 \asymp R \asymp D^{3/4}$, $A_2 \asymp 1$, with $U^2 \leq D \leq 7U^2/2$. The rows are the original unweighted nonprincipal rows, with fixed S, ν, u_0, their literal ray and capacity restrictions, and their primitive conductors $Q_u \asymp R$. Use one smooth complex W supported on $[1, 2]$, the full ordered cutoff, and the original raw zeros. A target bound means $O((DH)^{\epsilon}\mathcal{M}_{24}(W)^2 D^{3/4})$ for the specified contribution to the normalized pairing (129). Constants may depend on the fixed arithmetic data, scale constants and $\epsilon > 0$. Put $E = Nd$ and $e = \log_D E$. Here d is the intermediate divisor from the exact Möbius identity. All estimates include the original middle equality subtraction.

Theorem 47.1 (Full-frequency partner unit part)**.** *Take the unit part of C_d on the partner side of* (131)*, with all partner frequencies including the re-expressed direct term. A selection of the actual ordered pairs $mn = d$ may depend on d, m, n. Write $L_1 = U/Nm$, $L_2 = U/Nn$ and $l_s = \log_D \max(1, \min(L_1, L_2))$. The target holds over all $E \lesssim D^{1/6}$, and more generally on the dyads with $e \leq 1/4$ and $l_s \geq 1/4 + e/2$ at the level of powers. The remaining upper cost at a single dyad with $e \leq 1/4$, relative to R/E, has exponent $(1/4 + e/2 - l_s)_+/6$, where $x_+ = \max(0, x)$. The all-divisor bound for this component has power* $37/48$*.*

The low-divisor estimate uses the whole wide product support, even for balanced pairs with bounded nonunit d. Its gain comes from the forced radical of the actual product ideal in the joint positive norm. For instance, at $e = 1/4$ with the longer length of order $D^{1/2}$, the upper comparison still costs $D^{1/48}$ above R/E. This excess describes the estimate.

Theorem 47.2 (Selected partner frequencies)**.** *Let $T \leq N\mathfrak{k} < 2T$ be an actual partner frequency dyad and write $t = \log_D T$. A selector of modulus at most one may depend on $\mathfrak{k}$, the actual equality allocation and the fixed partner labels. It is independent of the row and the two varying partner*

cutoff ideals. In the actual Poisson representation of the unit part, including the direct term and the corresponding selected middle subtraction, the target holds in the following regions:

$$\begin{array}{c|l} 0 \le e \le 1/8 & t \le 15/32 + 3e/8 \\ 1/8 \le e \le 1/4 & t \le 33/64 \\ 1/4 \le e \le 5/16 & t \le 17/64 + e \\ 5/16 \le e < 27/56 & t \le 11/32 + 3e/4 \\ e \ge 27/56 & \textit{all frequency lengths.} \end{array} \tag{133}$$

Thus $T \lesssim D^{15/32}$ is sufficient over all intermediate divisors. Constant factors at the endpoints are allowed.

For selectors independent of the varying equality labels, further regions follow by grouping frequencies with prescribed ideal parts. If the rational content of $\mathfrak{k}$ is at least D^b, with $0 \le b \le 9/28$, then $T \lesssim D^{9/20+3b/5}$ suffices across all d. Here the content is the largest positive integer prime to the rational primes under S whose principal ideal divides $\mathfrak{k}$. At $E \gtrsim D^{1/4}$, content of order at least $D^{41/144}$ permits $T \lesssim D^{2/3}$. A good sixth-power base of norm at least of order $D^{1/16}$, or an S-part of norm at least of order $D^{3/10}$, suffices at every frequency length. These are conditions on the actual frequency, with all ideal labels and raw zeros retained.

Theorem 47.3 (Equality-wide correction)**.** *In the equality-side wide correction, write the equal-norm pair as $z_0 = g\ell,\ k = g\bar{\ell}$. Let ℓ_{g} retain the prime ideals of ℓ whose conjugates also avoid S. Against the full original partner complement, or its full-frequency step or wide part with its own subtraction, the target holds on $N\ell_{\mathrm{g}} \gtrsim D^{\theta(e)}$, where*

$$\theta(e) = \begin{cases} -17/12 + 37e/6, & 17/74 \le e \le 1/4, \\ 11/24 - 4e/3, & 1/4 \le e \le 11/32, \\ 0, & \textit{otherwise.} \end{cases}$$

In the distinct pure comparable selected rectangles $B, K \asymp R$ of the preceding wide analysis, the equality-side prefix extends to $Nd \lesssim D^{111/476}$, with the original allocation tests and the same choices of partner coefficient.

The small-conjugate-support band, unbalanced middle divisors and small-part main frequency range leave contributions to the coherent signed problem. The full signed assembly and the independent determinant and zero-location arguments remain open.

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